

Black Hole Evaporation. A Survey

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Abstract

This thesis is a review of black hole evaporation with emphasis on recent results obtained for two dimensional black holes. First, the geometry of the most general stationary black hole in four dimensions is described and some classical quantities are defined. Then, a derivation of the spectrum of the radiation emitted during the evaporation is presented. In section four, a two dimensional model which has black hole solutions is introduced, the so-called CGHS model. These two dimensional black holes are found to evaporate. Unlike the four dimensional case, the evaporation process can be studied analytically as long as the mass of the black hole is well above the two dimensional analog of the Planck mass. Finally, some proposals for resolving the so-called information paradox are reviewed and it is concluded that none of them is fully satisfactory.

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1 Introduction

In this work, we will review some of the known classical and quantum properties of black holes, with special emphasis on two dimensional black holes.

The general theory of relativity shows that if a star is sufficiently massive, no force can counterbalance its gravitational force so that it will collapse under its own weight. This theory further predicts that the endpoint of the collapse is a black hole, which is an object from which nothing can come out, at least if quantum effects are neglected. Some classical properties of four dimensional black holes are reviewed in section 2.

In 1975, Hawking discovered that black holes are not black: if quantum effects are taken into account, the black hole is found to emit particles with a thermal spectrum. In Hawking's calculation, which is described in section 3, the gravitational field of the black hole is treated classically but the matter radiated away is assumed to propagate in the fixed classical spacetime geometry of the black hole by the laws of quantum mechanics.

Since the outgoing Hawking radiation is a form of energy, it should modify the gravitational field of the black hole; this effect is called the back reaction of the radiation on the geometry. In particular, the back reaction should cause the mass of the black hole to decrease, the mass of the black hole being a property of the spacetime geometry. The effect of the back reaction is very difficult to analyze in $3 + 1$ dimensions; therefore, one is tempted to simplify the problem by going to $1 + 1$ dimensions.

Such a two dimensional model was proposed in 1991 by Callan, Giddings, Harvey and Strominger. This so-called CGHS model is the subject of section 4.

At the classical level, the CGHS model has black hole solutions (see section 4.1). Furthermore, the effect of back reaction can be studied at the (matter) one-loop level (section 4.3). One finds that the mass of the black hole decreases, as expected. Indeed, in one particular version of the CGHS model, the so-called RST model, proposed by Russo, Susskind and Thorlacius, the black hole is found to evaporate completely (section 4.3).

However, the main motivation for studying two dimensional models was to resolve the problem of information loss: Hawking discovered that the information contained in the matter which formed the black hole cannot be encoded in the outgoing radiation because the radiation is purely thermal. Therefore, most of the information is stored in the black hole. If the black hole evaporates completely, Hawking's calculation suggests that the information content will be lost

and this implies that the evolution of the black hole is not described by a unitary S -matrix; therefore, quantum mechanics would not apply to black holes.

The hope was therefore raised that the two dimensional models would resolve the issue of information loss but unfortunately, no definite conclusions could be drawn.

Several proposals have been made to resolve the information paradox in four dimensions, but as we will see in the conclusions, none of them is fully satisfactory today.

2 Classical Black Holes

In this section, we describe the geometry of the most general stationary black hole, the so-called Kerr black hole; we also define some useful quantities and finally, we derive some of its thermodynamical properties. More detailed reviews of classical black holes can be found in [1, chapters 6 and 12] and [2, chapters 31-34].

The interaction between the gravitational and the electromagnetic field is governed by the coupled Einstein-Maxwell equations. In the Lorentz gauge $\nabla^a A_a = 0$, they are:

$$\begin{cases} \nabla^a \nabla_a A_b - R_b^{d} A_d &= 0 \\ R_{ab} - \frac{1}{2} g_{ab} R &= 8\pi T_{ab} , \end{cases}$$

where T_{ab} is the electromagnetic stress-energy tensor:

$$T_{ab} = \frac{1}{4\pi} \{ F_{ac} F_b^{c} - \frac{1}{4} g_{ab} F_{de} F^{de} \} .$$

The general stationary black hole solution subject to the constraints that the mass, angular momentum and electric charge of the black hole take definite values, is described by the Kerr metric

$$\begin{aligned} ds^2 = & - \left(\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\varphi \\ & + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 , \end{aligned} \quad (1)$$

and the electromagnetic potential

$$A_a = -\frac{Qr}{\Sigma} \left((dt)_a - a \sin^2 \theta (d\varphi)_a \right) , \quad (2)$$

where

$$\begin{cases} \Sigma &= r^2 + a^2 \cos^2 \theta \\ \Delta &= r^2 + a^2 + Q^2 - 2Mr . \end{cases} \quad (3)$$

Q , M , a are the three parameters of the family of solutions and can be verified to be the electric charge, the mass and the angular momentum per unit mass of the black hole, respectively.

When $a = 0$, the metric reduces to the Reissner-Nordstrom metric

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 \\ & + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \end{aligned} \quad (4)$$

and when $a = Q = 0$, we recover the Scharzschild metric

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \quad (5)$$

which for symplicity we will analyze below.

We define

$$r^* = r + 2M \ln \left(\frac{r}{2M} - 1 \right) ,$$

so that

$$ds^2 = - \left(1 - \frac{2M}{r}\right) (dt^2 - dr^{*2}) + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) .$$

Note that $r = 2M$ and $r = +\infty$ correspond to $r^* = -\infty$ resp $r^* = +\infty$.

Introducing the null coordinates $\begin{Bmatrix} u \\ v \end{Bmatrix} = t \mp r^*$, we get (suppressing the angular dependence)

$$\begin{aligned} ds^2 &= - \left(1 - \frac{2M}{r}\right) du dv \\ &= - \frac{2M}{r} e^{-\frac{r}{2M}} e^{\frac{v-u}{4M}} du dv . \end{aligned}$$

Defining the null coordinates

$$\begin{cases} U = -e^{-\frac{u}{4M}} & -\infty < U < 0 \\ V = e^{\frac{v}{4M}} & 0 < V < \infty , \end{cases}$$

the metric takes the form

$$\begin{aligned} ds^2 &= - \frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV \\ &= \frac{32M^3}{r} e^{-\frac{r}{2M}} (-dT^2 + dX^2) , \end{aligned}$$

where

$$\begin{cases} U = T - X \\ V = T + X . \end{cases}$$

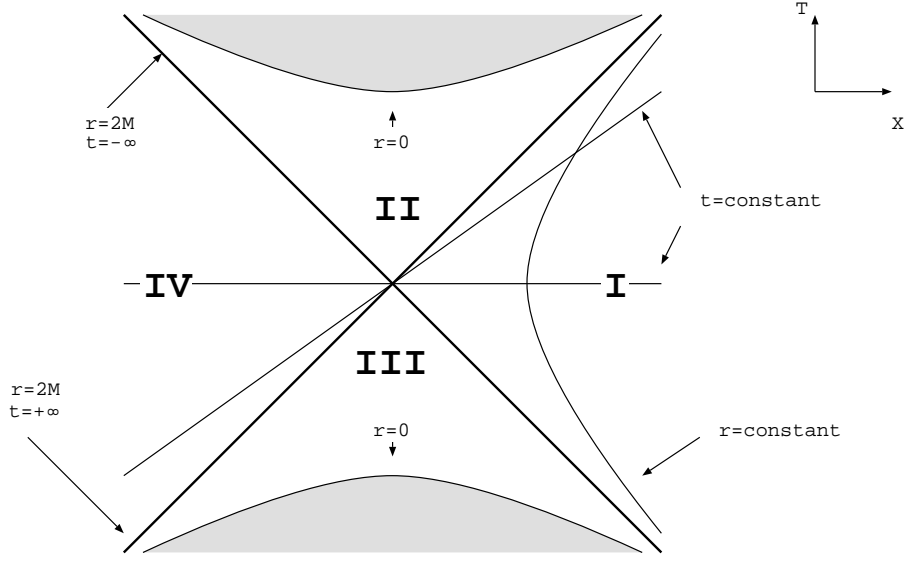


Figure 1:

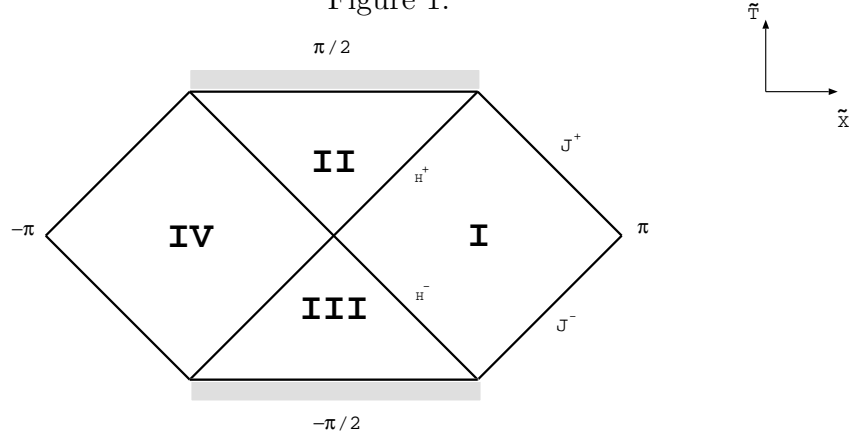


Figure 2:

The relation between (t, r) and (T, X) is

$$\begin{cases} X^2 - T^2 = -UV = e^{\frac{v-u}{4M}} = e^{\frac{r^*}{2M}} = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \\ \frac{X+T}{X-T} = \frac{V}{-U} = e^{\frac{u+v}{4M}} = e^{\frac{t}{2M}}, \end{cases}$$

or

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = X^2 - T^2,$$

and

$$\frac{t}{2M} = 2 \arctan \frac{T}{X} = \ln \frac{1 + \frac{T}{X}}{1 - \frac{T}{X}}.$$

In these coordinates, the singularity in the metric components at $r = 2M$ has disappeared and as a consequence, the spacetime can be extended by allowing the ranges of U and V to be unrestricted. The resulting maximally extended

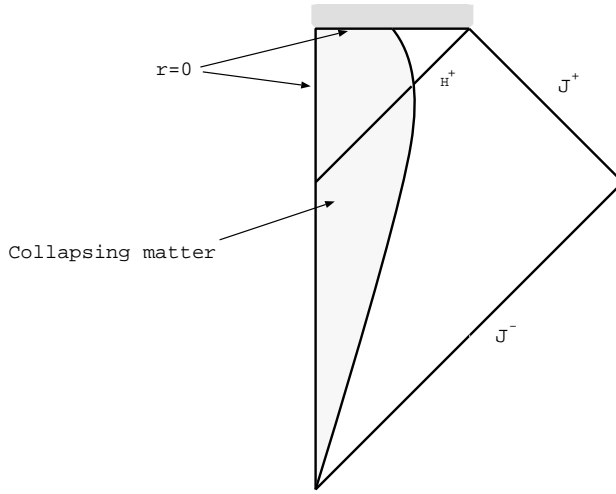


Figure 3:

Schwarzschild spacetime is depicted in fig. 1.

Another representation of this spacetime can be obtained by mapping U, V to null coordinates whose ranges are restricted to finite intervals, for example

$$\begin{cases} U = \tan \frac{1}{2} \tilde{U} \\ V = \tan \frac{1}{2} \tilde{V} . \end{cases}$$

The resulting "conformal" or "Penrose" diagram is shown in fig. 2.

The most salient feature of the maximally extended Schwarzschild spacetime is that it has a future event horizon H^+ , i.e., a null hypersurface from behind which it is impossible to escape to future null infinity J^+ without exceeding the speed of light. It also has a past event horizon H^- , i.e., a null hypersurface behind which it is impossible to go when starting from past null infinity J^- .

Both the past and the future event horizon are located at $r = 2M$, see fig. 1.

The singularity at $r = 0$ is a true physical singularity— for example, $R_{abcd}R^{abcd}$ is infinite there. However, thanks to the horizons, it cannot be seen from outside the black hole, that is, from region *I* in fig. 1 and fig. 2.

For a collapsing star, only part of region *I* and *II* will be generated (see fig. 3).

As for the general Kerr black hole, it has been proved that it is the only possible stationary vacuum black hole, i.e., a classical black hole is characterized uniquely by its mass, angular momentum and electric charge: a black hole has no

hair. To show this in a special case, we try to find the stationary solutions of the Klein-Gordon equation in the Schwarzschild metric. Decomposing the solution in spherical harmonics, we will show later that the radial part $f_{l,m}(r)$ satisfies

$$\partial_{r^*}^2 f_{l,m}(r) = \left(1 - \frac{2M}{r}\right) \left(m^2 + \frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) f_{l,m}(r) . \quad (6)$$

Because the second derivative with respect to r^* is strictly positive outside the black hole, a solution which vanishes exponentially at infinity must blow up at the horizon since $r^* \rightarrow -\infty$ there. Thus, there is no physically acceptable stationary solution to the Klein-Gordon equation, and a Schwarzschild black hole has no Klein-Gordon charge.

From (1), we see that the Kerr metric is singular where $\Sigma = r^2 + a^2 \cos^2 \theta = 0$ and where $\Delta = r^2 + a^2 + Q^2 - 2Mr = 0$.

Evaluation of curvature invariants such as $R_{abcd}R^{abcd}$ shows that the singularity at $\Sigma = 0$ is a real singularity when $M \neq 0$, which cannot be removed by extending the manifold.

When $Q^2 + a^2 > M^2$, there are no solutions to the equation $\Delta = 0$: we are left with a "naked" singularity at $\Sigma = 0$, so in this case the Kerr solution is not physically acceptable.

However, when $Q^2 + a^2 \leq M^2$, Δ vanishes at $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$. It has been shown that the latter singularities (at $r = r_{\pm}$) are of the same nature as the $r = 2M$ singularity in the Schwarzschild spacetime. In particular, the horizon at $r = r_+$ prevents an external observer from seeing the real singularity at $\Sigma = 0$.

A conformal diagram of the extended charged Kerr spacetime with $a \neq 0$ is shown in fig. 4, for the non-extreme case $Q^2 + a^2 < M^2$.

In a physically realistic gravitational collapse, the spacetime is expected to be qualitatively similar to that depicted in fig. 3 because of instabilities associated with the so-called Cauchy horizon at $r = r_-$.

For a rotating black hole, $a \neq 0$, the Killing field χ^a which is a null generator of the horizon does not coincide with the time translation Killing field $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$. The other Killing field available being $\psi^a = \left(\frac{\partial}{\partial \varphi}\right)^a$, we must have

$$\chi^a = \xi^a + \Omega_H \psi^a .$$

So the event horizon is rotating with angular velocity Ω_H . It can be calculated

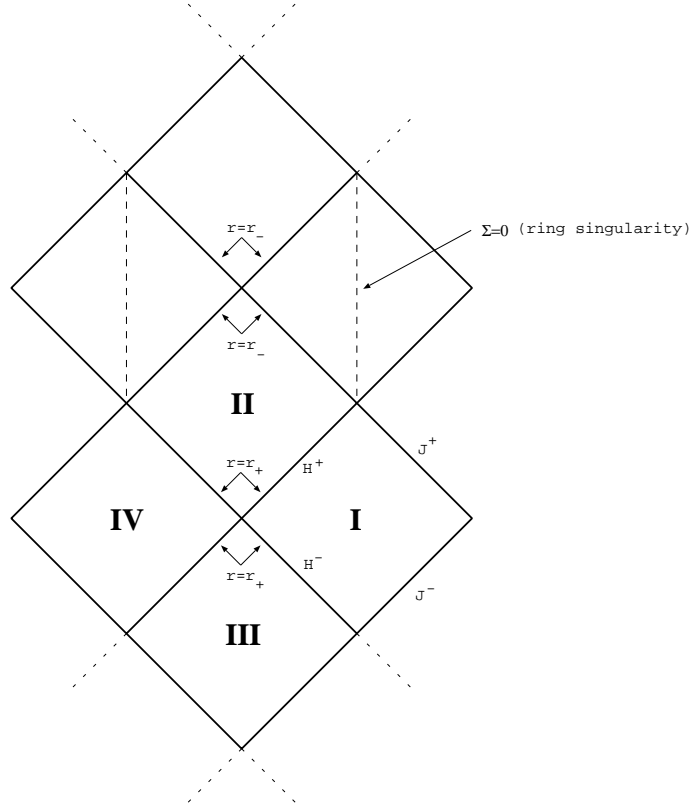


Figure 4:

by requiring χ^a to be null on the horizon

$$\begin{aligned}
0 = \chi^a \chi_a &= \xi^a \xi_a + 2 \xi^a \psi_a \Omega_H + \psi^a \psi_a \Omega_H^2 \\
&= \frac{a^2 \sin^2 \theta}{\Sigma} - \frac{2a \sin^2 \theta (r_+^2 + a^2)}{\Sigma} \Omega_H + \frac{(r_+^2 + a^2)^2}{\Sigma} \sin^2 \theta \Omega_H^2 .
\end{aligned}$$

So

$$\left((r_+^2 + a^2) \Omega_H - a \right)^2 = 0 \Leftrightarrow \Omega_H = \frac{a}{(r_+^2 + a^2)} . \quad (7)$$

Another important quantity is the surface gravity of the black hole κ . For a static non-rotating black hole, it can be defined as the force exerted by a stationary observer at infinity on a stationary particle with unit mass (the particle can be thought of as being connected to the observer by means of a long massless string).

The four-velocity of the particle is

$$u^a = (-\xi^a \xi_a)^{-\frac{1}{2}} \xi^a \equiv \frac{1}{V} \xi^a ,$$

where $V^2 = -\xi^a \xi_a$ is the redshift factor, so the acceleration is

$$\begin{aligned} a^b &= u^a \nabla_a u^b = \frac{1}{V} \xi^a \nabla_a \frac{1}{V} \xi^b = \frac{1}{V^2} \xi^a \nabla_a \xi^b \\ &= -\frac{1}{V^2} \xi^a \nabla^b \xi_a = \frac{1}{2} \frac{1}{V^2} \nabla^b (-\xi^a \xi_a) \\ &= \frac{1}{2} \frac{1}{V^2} \nabla^b V^2 = \nabla^b \ln V , \end{aligned}$$

and the magnitude of the local force per unit mass is

$$\sqrt{a^c a_c} = \frac{1}{V} \sqrt{(\nabla_a V \nabla^a V)} .$$

a goes to infinity as one approaches the horizon because the redshift factor $V^2 = -\xi^a \xi_a = -g_{tt}$ goes to zero.

The force at infinity can be defined as

$$a_\infty^b = -\nabla^b e_\infty ,$$

where e_∞ is the energy per unit mass of the stationary particle as measured by the stationary observer at infinity

$$e_\infty = -\xi^a u_a = -\xi^a \frac{1}{V} \xi_a = V .$$

We see that

$$a_\infty = V a = \sqrt{(\nabla_a V \nabla^a V)} .$$

For a Schwarzschild black hole

$$V^2 = -\xi^a \xi_a = -g_{tt} = \left(1 - \frac{2M}{r}\right) ,$$

and

$$\begin{aligned} \kappa &= \lim_{r \rightarrow 2M} a_\infty = \lim_{r \rightarrow 2M} \sqrt{g^{rr} (\partial_r V)^2} = \lim_{r \rightarrow 2M} \sqrt{g^{rr}} \partial_r V \\ &= \lim_{r \rightarrow 2M} \frac{1}{\sqrt{g_{rr}}} \frac{1}{2V} \partial_r V^2 = \frac{1}{2} \partial_r V^2 \Big|_{r=2M} = \frac{1}{2} \frac{2M}{r^2} \Big|_{r=2M} \\ &= \frac{1}{4M} . \end{aligned} \tag{8}$$

For a rotating black hole, we define a^b by

$$a^b = \frac{\chi^a \nabla_a \chi^b}{-\chi^a \chi_a} = \nabla^b \ln V ,$$

where now

$$\begin{aligned} V^2 = -\chi^a \chi_a &= -(\xi^a + \Omega_H \psi^a)(\xi_a + \Omega_H \psi_a) \\ &= -g_{tt} - 2\Omega_H g_{t\varphi} - \Omega_H^2 g_{\varphi\varphi} \\ &= \frac{1}{\Sigma} \left\{ \Delta - a^2 \sin^2 \theta + 2a \sin^2 \theta (r^2 + a^2 - \Delta) \Omega_H \right. \\ &\quad \left. - \left((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \right) \sin^2 \theta \Omega_H^2 \right\} . \end{aligned}$$

It can be shown [1, chapter 12] that κ is constant over the event horizon so we can evaluate it at $\theta = 0$. Using $\partial_\theta V(r, \theta = 0) = 0$, we get

$$\begin{aligned} \kappa &= \sqrt{g^{rr}(\partial_r V)^2} = \sqrt{\frac{\Delta}{\Sigma}} \frac{1}{2V} \partial_r V^2 = \frac{1}{2} \partial_r \left(\frac{\Delta}{\Sigma} \right) \Big|_{r=r_+, \theta=0} \\ &= \frac{1}{2} \frac{\partial_r \Delta|_{r=r_+}}{r_+^2 + a^2} = \frac{1}{2} \frac{r_+ - r_-}{r_+^2 + a^2} \\ &= \frac{\sqrt{M^2 - a^2 - Q^2}}{2M(M + \sqrt{M^2 - a^2 - Q^2}) - Q^2} . \end{aligned} \tag{9}$$

κ appears in the relation between the Killing parameter v along the null generator of the horizon defined by $\chi^a \nabla_a v = 1$ and the affine parameter λ along the same generator: it can be proved [1, chapter 12] that

$$\lambda \propto e^{\kappa v} .$$

This shows that the null coordinate V that we defined earlier when we analyzed the Schwarzschild metric is an affine parameter along the future horizon because we had $V = e^{\frac{v}{4M}} = e^{\kappa v}$.

The area A of the event horizon of a Kerr black hole is

$$\begin{aligned} A &= \int_{r=r_+} d\theta d\varphi \sqrt{g_{\theta\theta} g_{\varphi\varphi}} \\ &= \int_{r=r_+} d\theta d\varphi \sqrt{\Sigma \frac{(r_+^2 + a^2)^2 \sin^2 \theta}{\Sigma}} \\ &= 4\pi (2Mr_+ - Q^2) . \end{aligned}$$

Clearly, the area of the event horizon is constant in time. This is a special case of the black hole area theorem which Hawking proved in 1971 by using global properties of spacetime: it states that the total area of all black holes in the universe cannot decrease, $\delta A \geq 0$. The area of a black hole is thus analogous to the entropy of a thermodynamical system, and in fact, the area theorem is sometimes called the second law of black hole dynamics.

There is also an analogous first law which we now derive by studying the variation of A with respect to M and $J \equiv Ma$

$$\frac{1}{8\pi} \delta A = r_+ \delta M + M \delta r_+ .$$

Now

$$\begin{aligned} \delta r_+ &= \delta \left(M + \sqrt{M^2 - a^2 - Q^2} \right) \\ &= \delta M + \frac{1}{2} (M^2 - a^2 - Q^2)^{-\frac{1}{2}} 2 (M \delta M - a \delta a) \\ &= \delta M + (r_+ - M)^{-1} (M \delta M - a \delta a) , \\ a \delta a &= a \delta (J/M) = \frac{a}{M} \delta J - \frac{a^2}{M} \delta M . \end{aligned}$$

So

$$\begin{aligned} \frac{1}{8\pi} \delta A &= \left\{ r_+ + M \left(1 + (r_+ - M)^{-1} \left(\frac{M^2 + a^2}{M} \right) \right) \right\} \delta M - \frac{a}{(r_+ - M)} \delta J \\ &= \frac{r_+^2 + a^2}{r_+ - M} \delta M - \frac{a}{r_+ - M} \delta J = \frac{1}{\kappa} \delta M - \frac{\Omega_H}{\kappa} \delta J , \end{aligned}$$

where we have used $\kappa = \frac{r_+ - M}{r_+^2 + a^2}$ and $\Omega_H = \frac{a}{r_+^2 + a^2}$.

Thus

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J . \quad (10)$$

This suggests that the laws of black hole dynamics are really the laws of thermodynamics applied to black holes. In this case, the temperature and entropy of the black hole are $T = \alpha \kappa$ and $S = \frac{1}{8\pi\alpha} A$, respectively. However, for classical black holes, the temperature is zero because a classical black hole only absorbs energy without emitting anything. This would be in contradiction with the conjecture made above. When quantum effects are taken into account, this paradox is resolved because, as was first shown by Hawking, a black hole emits radiation like

a blackbody at temperature $\hbar\kappa/2\pi$ (see next section), and therefore, the entropy of a black hole is $S = \hbar^{-1}A/4$.

3 Hawking Radiation

The black hole radiance was found [3] by studying the propagation of quantized, noninteracting matter fields in the classical geometry of the black hole.

The simplest example one would consider is a neutral spin zero particle described by the real Klein-Gordon field which propagates in region I of the extended Schwarzschild spacetime, fig. 2.

It satisfies the equation

$$\square\phi - m^2\phi = 0 ,$$

where

$$\square\phi = g^{ab}\nabla_a\nabla_b\phi = (-g)^{-\frac{1}{2}}\partial_a\left((-g)^{\frac{1}{2}}g^{ab}\partial_b\phi\right) .$$

Here $-g_{tt} = g_{r^*r^*} = \left(1 - \frac{2M}{r}\right)$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2\sin^2\theta$ and all non-diagonal components are zero.

Also $\sqrt{-g} = \left(1 - \frac{2M}{r}\right)r^2\sin\theta$, so

$$\begin{aligned} 0 = (\square - m^2)\phi &= g^{00}\partial_0^2\phi + (-g)^{-\frac{1}{2}}\partial_{r^*}\left((-g)^{\frac{1}{2}}g^{r^*r^*}\partial_{r^*}\phi\right) \\ &+ \frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\phi\right) - m^2\phi . \end{aligned}$$

Setting $\phi = \frac{1}{r}f(r, t)Y_{lm}(\theta, \varphi)$, the equation becomes:

$$\begin{aligned} &-\left(1 - \frac{2M}{r}\right)^{-1}\partial_t^2\frac{f}{r} + \left(1 - \frac{2M}{r}\right)^{-1}r^{-2}\partial_{r^*}\left(\left(1 - \frac{2M}{r}\right)r^2\left(1 - \frac{2M}{r}\right)^{-1}\partial_{r^*}\frac{f}{r}\right) \\ &-\frac{l(l+1)}{r^2}\frac{f}{r} - m^2\frac{f}{r} = 0 , \end{aligned}$$

or

$$\partial_t^2 f - r^{-1}\partial_{r^*}\left(r^2\partial_{r^*}\frac{f}{r}\right) + \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + m^2\right)f = 0 .$$

Now

$$\begin{aligned} \partial_{r^*}\left(r^2\partial_{r^*}\frac{f}{r}\right) &= \partial_{r^*}\left(r\partial_{r^*}f + r^2f\partial_{r^*}\frac{1}{r}\right) \\ &= r\partial_{r^*}^2f + \partial_{r^*}\left(r^2\partial_{r^*}\frac{1}{r}\right)f = r\partial_{r^*}^2f + \left(1 - \frac{2M}{r}\right)\partial_r\left(r^2\left(1 - \frac{2M}{r}\right)\frac{-1}{r^2}\right)f \\ &= r\partial_{r^*}^2f - \left(1 - \frac{2M}{r}\right)\frac{2M}{r^2}f , \end{aligned}$$

so we end up with the equation:

$$\partial_t^2 f - \partial_{r^*}^2 f + \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right) f = 0 . \quad (11)$$

This equation has the form of a wave equation for a massless scalar field in a flat two dimensional spacetime with a scalar potential

$$V(r^*) = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) ,$$

where we have specialized to $m = 0$.

As $r^* \rightarrow -\infty$ or $r \rightarrow 2M$, $V(r^*) \sim \left(1 - \frac{2M}{r}\right) \sim e^{\frac{r^*}{2M}}$, i.e., the potential vanishes exponentially.

As $r^* \rightarrow +\infty$ or $r \rightarrow +\infty$, the potential vanishes at least as $\frac{1}{r^{*2}}$.

Thus, in the asymptotic past, any solution reduces to the sum of two free wave packets, one coming from the white hole horizon H^- , called $f_-(u)$, and one coming from past null infinity J^- , called $g_-(v)$.

Similarly, in the asymptotic future, any solution reduces to the sum of two free wave packets: one going towards the black hole horizon H^+ , $g_+(v)$, and one going towards future null infinity J^+ , $f_+(u)$.

Any incoming one particle state is thus the linear combination of a one particle state coming from H^- and a one particle state coming from J^-

$$\mathcal{H}_{\mathcal{Y}} = \mathcal{H}_{\mathcal{H}^-} \oplus \mathcal{H}_{\mathcal{J}^-} .$$

$\mathcal{H}_{\mathcal{J}^-}$ is defined as the vector space of all solutions $g_-(v)$ containing only positive frequencies with respect to some null coordinate $\tilde{v}(v)$

$$g_-^{(+)}(\tilde{v}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega e^{-i\omega\tilde{v}} \hat{g}_-^{(+)}(\omega) .$$

Here $\tilde{v} = v = t + r^*$ is the natural choice because J^- is an asymptotically flat region and in this region, (u, v) are minkowskian (null) coordinates, $ds^2 = -du dv$.

However, there is an ambiguity in the definition of $\mathcal{H}_{\mathcal{H}^-}$ because H^- is *not* asymptotically flat and as a consequence, there is no unique choice of $\tilde{u}(u)$: we could

choose it to be either the Killing parameter $u = t - r^*$ or the affine parameter $U = -e^{-\frac{u}{4M}}$.

For the collapsing spherical body of fig. 3, the spacetime contains no white hole horizon so $\mathcal{H}_{\gamma\backslash}$ is just $\mathcal{H}_{\mathcal{J}^-}$ and is therefore uniquely defined.

However, an ambiguity still remains in the definition of $\mathcal{H}_{\gamma\sqcup} = \mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+}$ because the definition of positive frequency solutions on the black hole horizon is not unique.

We assume that the Fock space \mathcal{F} of the quantum field $\hat{\phi}$ is isomorphic to the Fock space $\mathcal{F}_{\gamma\backslash}(\mathcal{H}_{\gamma\backslash})$ of incoming particles, where $\mathcal{F}_{\gamma\backslash}(\mathcal{H}_{\gamma\backslash})$ is the direct sum of the Hilbert spaces of no particle states, one particle states, two particle states and so on. Let $U : \mathcal{F} \rightarrow \mathcal{F}_{\gamma\backslash}$ denote this isomorphism.

If we define

$$U\hat{\phi}(x)U^{-1} = \sum_{i=1}^{\infty} \sigma_i(x) a_{in}(\bar{\sigma}_i) + \bar{\sigma}_i(x) a_{in}^{\dagger}(\sigma_i) ,$$

where $\{\sigma_i(x)\}$ is an orthonormal basis of $\mathcal{H}_{\gamma\backslash}$ for x in the asymptotic past J^- , we see that for these x 's, $U\hat{\phi}(x)U^{-1}$ is the usual free field operator in flat spacetime.

In the same way, we denote by W the isomorphism between \mathcal{F} and $\mathcal{F}_{\gamma\sqcup}$ and we define

$$W\hat{\phi}(x)W^{-1} = \sum_{i=1}^{\infty} \rho_i(x) a_{out}(\bar{\rho}_i) + \bar{\rho}_i(x) a_{out}^{\dagger}(\rho_i) ,$$

where $\{\rho_i(x)\}$ is an orthonormal basis of $\mathcal{H}_{\gamma\sqcup}$ for x in the asymptotic future. $W\hat{\phi}(x)W^{-1}$ then reduces to the free field operator in flat spacetime.

So

$$\begin{aligned} & S \left(\sum_{j=1}^{\infty} \sigma_j a_{in}(\bar{\sigma}_j) + \bar{\sigma}_j a_{in}^{\dagger}(\sigma_j) \right) S^{-1} \\ &= \sum_{j=1}^{\infty} \rho_j a_{out}(\bar{\rho}_j) + \bar{\rho}_j a_{out}^{\dagger}(\rho_j) , \end{aligned} \tag{12}$$

where $S = WU^{-1}$.

Taking the Klein-Gordon inner product [4] of this equation with σ_i , the left hand side becomes

$$\begin{aligned} & \left(\sigma_i, \sum_{j=1}^{\infty} \sigma_j S a_{in}(\bar{\sigma}_j) S^{-1} + \bar{\sigma}_j S a_{in}^{\dagger}(\sigma_j) S^{-1} \right)_{K.G.} \\ &= S a_{in}(\bar{\sigma}_i) S^{-1} . \end{aligned}$$

In the asymptotic future, $\sigma_i(x)$ reduces to a solution of the Klein-Gordon equation in flat spacetime and contains both a positive and a negative frequency part, denoted by μ and λ , respectively.

Define the operators $C : \mathcal{H}_{\setminus} \rightarrow \mathcal{H}_{\sqcup}$ and $D : \mathcal{H}_{\setminus} \rightarrow \overline{\mathcal{H}}_{\sqcup}$ in such a way that $C\sigma = \mu$ and $D\sigma = \lambda$.

The Klein-Gordon product of the right hand side of (12) with $\sigma_i(x)$ is

$$\begin{aligned} & \sum_{i=1}^{\infty} (\sigma_i, \rho_j)_{K.G.} a_{out}(\bar{\rho}_j) + (\sigma_i, \bar{\rho}_j)_{K.G.} a_{out}^{\dagger}(\rho_j) \\ &= a_{out}(\overline{C\sigma_i}) - a_{out}^{\dagger}(\overline{D\sigma_i}) , \end{aligned}$$

so we have

$$S a_{in}(\bar{\sigma}_i) S^{-1} = a_{out}(\overline{C\sigma_i}) - a_{out}^{\dagger}(\overline{D\sigma_i}) . \quad (13)$$

This relation defines the action of the S-matrix. It can be shown that S is unitary.

Since $\mathcal{F}_{\sqcup}(\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+})$ is isomorphic to $\mathcal{F}_{\mathcal{H}^+}(\mathcal{H}_{\mathcal{H}^+}) \otimes \mathcal{F}_{\mathcal{J}^+}(\mathcal{H}_{\mathcal{J}^+})$, any $|\psi\rangle \in \mathcal{F}_{\sqcup}$ can be written as:

$$|\psi\rangle = c_{hj} |h\rangle |j\rangle ,$$

where $c_{hj} = \langle h, j | \psi \rangle$, and $|h\rangle$ and $|j\rangle$ are orthonormal bases in $\mathcal{F}_{\mathcal{H}^+}$ resp. $\mathcal{F}_{\mathcal{J}^+}$.

If O is an operator acting on $\mathcal{F}_{\mathcal{J}^+}$, its expectation value in the state $|\psi\rangle$ is

$$\begin{aligned} \langle \psi | O | \psi \rangle &= \langle h', j' | \bar{c}_{h'j'} O c_{hj} | h, j \rangle = c_{hj} \bar{c}_{h'j'} \langle h' | h \rangle \langle j' | O | j \rangle \\ &= c_{hj} \bar{c}_{h'j'} \langle j' | O | j \rangle = tr_{\mathcal{F}_{\mathcal{J}^+}} \{ c_{hj} \bar{c}_{h'j'} | j \rangle \langle j' | O \} . \end{aligned}$$

Therefore, a pure state $|\psi\rangle$ in $\mathcal{F}_{\mathcal{H}^+} \otimes \mathcal{F}_{\mathcal{J}^+}$ is viewed as a mixed state in $\mathcal{F}_{\mathcal{J}^+}$ described by the density matrix $\rho = \langle h, j | \psi \rangle \langle \psi | h, j' \rangle | j \rangle \langle j' |$

$$\langle \psi | O | \psi \rangle = tr_{\mathcal{F}_{\mathcal{J}^+}} \{ \rho O \} . \quad (14)$$

A redefinition of the notion of positive frequency on the black hole horizon induces a unitary transformation, of the form given by (13), acting on $\mathcal{F}_{\mathcal{H}^+}$

$$|\psi'\rangle = S |\psi\rangle = c_{hj} (S |h\rangle) |j\rangle .$$

But since $\langle h' | S^\dagger S | h \rangle = \langle h' | h \rangle$, ρ is left unchanged which means that at J^+ , one obtains unambiguous physical predictions: the results of measurements performed at J^+ do not depend on the choice of positive frequency on the black hole horizon.

Coming back to the fundamental equation (13),

$$S a_{in}(\bar{\sigma}_i) S^{-1} = a_{out}(\overline{C\sigma_i}) - a_{out}^\dagger(\overline{D\sigma_i}) ,$$

we can solve for $|\psi\rangle \equiv S|0_{in}\rangle$, where $|0_{in}\rangle$ is the vacuum state of $\mathcal{F}_{\setminus\lambda}$: we write

$$\psi = (c, \eta^{a_1}, \eta^{a_1 a_2}, \eta^{a_1 a_2 a_3}, \dots) ,$$

where $\eta^{a_1 \dots a_n}$ is an n -particle state of $\mathcal{F}_{\sqcap\sqcup}(\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+})$, i.e.,

$$\eta^{a_1 \dots a_n} \in \bigotimes_S^n (\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+}) ,$$

and apply both sides of (13) to $|\psi\rangle$

$$S a_{in}(\bar{\sigma}_i) |0_{in}\rangle = 0 = \{a_{out}(\overline{C\sigma_i}) - a_{out}^\dagger(\overline{D\sigma_i})\} |\psi\rangle ,$$

or

$$0 = (a_{out}(\bar{\tau}_i) - a_{out}^\dagger(E\bar{\tau}_i)) |\psi\rangle , \quad (15)$$

where we have defined $\tau_i \equiv C\sigma_i$ and $E = \overline{D}\overline{C}^{-1}$.

Using the definition of a_{out} and a_{out}^\dagger , we obtain the equations [4]

$$\left\{ \begin{array}{l} \eta^a \bar{\tau}_a = 0 \\ \sqrt{2} \eta^{ab} \bar{\tau}_a = c (E\bar{\tau})^b \\ \sqrt{3} \eta^{abc} \bar{\tau}_a = \sqrt{2} (E\bar{\tau})^{(b} \eta^{c)} \\ \vdots \\ \vdots \\ \vdots \end{array} \right. , \quad \forall \tau^a .$$

The solution is

$$\left\{ \begin{array}{ll} \eta^{a_1 \dots a_n} = 0 & , \quad n \text{ odd} \\ \eta^{a_1 \dots a_n} = c \frac{(n!)^{\frac{1}{2}}}{2^{\frac{n}{2}} (\frac{n}{2})!} \varepsilon^{(a_1 a_2} \varepsilon^{a_3 a_4} \dots \varepsilon^{a_{n-1} a_n)} & , \quad n \text{ even} , \end{array} \right.$$

so

$$\psi = \psi(\varepsilon^{ab}) = c \left(1, 0, \sqrt{\frac{1}{2}} \varepsilon^{ab}, 0, \sqrt{\frac{3.1}{4.2}} \varepsilon^{(ab} \varepsilon^{cd)}, 0, \dots \right), \quad (16)$$

where ε^{ab} is the two particle state corresponding to E .

We observe that particle creation occurs only if $D \neq 0$ and that the particles are produced in pairs.

Let $P_{\omega lm}$ denote the solutions which at J^+ have the form,

$$\frac{1}{\sqrt{\omega}} \frac{e^{-i\omega u}}{r} Y_{l,m}(\theta, \varphi),$$

and construct the wave packets

$$P_{jnlm} = \frac{1}{\sqrt{E}} \int_{jE}^{(j+1)E} d\omega P_{\omega lm} e^{\frac{2\pi i n \omega}{E}},$$

where $0 < E \ll 1$.

These wave packets are made up of frequencies around $\omega = \left(j + \frac{1}{2}\right) E$ and are peaked around $u = \frac{2\pi n}{E}$ with a time spread $\Delta u = \frac{2\pi}{E}$.

At J^+ , they provide an orthonormal basis for $\mathcal{H}_{\mathcal{J}^+}$ and as elements of $\mathcal{H}_{\mathcal{J}^+}$, we denote them by ${}_i\rho^a$, where i stands for $jnlm$.

For an eternal black hole (not formed by gravitational collapse, fig. 2), we similarly define the wave packets Q_{jnlm} from the solutions that reduce to

$$\frac{1}{\sqrt{\omega}} \frac{e^{-i\omega v}}{r} Y_{l,m}(\theta, \varphi),$$

at H^+ .

In the case of a black hole formed by gravitational collapse, fig. 3, v is defined such that it agrees with the Killing parameter outside the collapsing matter. The ambiguity in defining v will not affect Q_{jnlm} for late times, i.e., for large n .

At H^+ , we choose the $Q_{jnlm} : s$ as part of our basis in $\mathcal{H}_{\mathcal{H}^+}$ and denote them by ${}_i\sigma^a$.

Together $\{{}_i\rho^a\}$ and $\{{}_i\sigma^a\}$ form a late time basis of $\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+}$.

Another late time basis in $\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+}$ can be constructed as follows: denote

by $X_{\omega lm}$ and $Y_{\omega lm}$ the solutions in the eternal black hole spacetime that have the form $\frac{1}{\sqrt{\omega}} \frac{e^{-i\omega u}}{r} Y_{l,m}(\theta, \varphi)$ at H^- resp. $\frac{1}{\sqrt{\omega}} \frac{e^{-i\omega v}}{r} Y_{l,m}(\theta, \varphi)$ at J^- , and form the corresponding wave packets, X_{jnlm} and Y_{jnlm} . We must have

$$X_{jnlm} = t_{lm}(\omega) P_{jnlm} + r_{lm}(\omega) Q_{jnlm} ,$$

and

$$Y_{jnlm} = T_{lm}(\omega) Q_{jnlm} + R_{lm}(\omega) P_{jnlm} ,$$

where t , T , and r , R are transmission resp. reflection amplitudes.

The $X_{jnlm} : s$ and the $Y_{jnlm} : s$ are thus elements of a new late time basis in $\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+}$ and we will call them ${}_i\lambda^a$ resp. ${}_i\gamma^a$. So

$${}_i\lambda^a = t_i {}_i\rho^a + r_i {}_i\sigma^a ,$$

and

$${}_i\gamma^a = T_i {}_i\sigma^a + R_i {}_i\rho^a .$$

ε^{ab} is determined by the action of the operator DC^{-1} on $\{{}_i\lambda^a, {}_i\gamma^a\}$: if we propagate the wave packet corresponding to ${}_i\gamma^a$ backwards in time from $H^+ \cup J^+$, it will be almost entirely scattered back to J^- by the static Schwarzschild geometry and the resulting wave packet will be the purely positive frequency wave packet Y_{jnlm} . Hence

$$DC^{-1} {}_i\gamma^a = 0 .$$

On the other hand, the wave packet corresponding to ${}_i\lambda^a$ is completely transmitted (backwards in time) to the surface of the collapsing body because in the extended Schwarzschild spacetime it would, by definition, be transmitted in its entirety to H^- . For an observer on the collapsing body, the wave packets with sufficiently high n will have the form

$$X_{jnlm} = \frac{1}{\sqrt{E}} \int_{jE}^{(j+1)E} d\omega \left\{ \frac{1}{\sqrt{\omega}} \frac{e^{-i\omega u}}{r} Y_{l,m}(\theta, \phi) \right\} e^{\frac{2\pi i n \omega}{E}} .$$

Because u goes to infinity on the event horizon, the surfaces of constant phase of $\phi(u) \equiv e^{-i\omega u}$ pile up close to it (see fig. 5). To the observer, the frequency thus appears to go to infinity as the radius of the body goes to $2M$.

Indeed, the dependence of the phase on the proper time (or affine parameter)

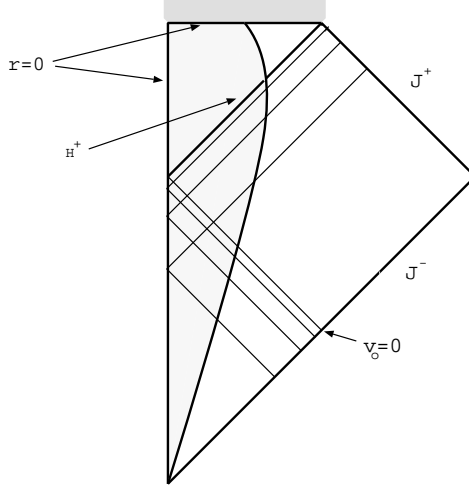


Figure 5:

λ along the geodesic of the observer or any other observer entering the black hole, is

$$+\omega u(U(\lambda)) = -\frac{\omega}{\kappa} \ln(-U(\lambda)) .$$

Since U depends smoothly on λ and satisfies $\frac{dU}{d\lambda} \neq 0$, we can write

$$U(\lambda) = \left(\frac{dU}{d\lambda} \right)_{\lambda=0} \lambda \quad , \text{ close to } H^+ ,$$

where λ is chosen to be zero at H^+ . Therefore, close to the horizon

$$\omega u = -\frac{\omega}{\kappa} \ln(-\alpha\lambda) ,$$

where $\alpha \equiv \left(\frac{dU}{d\lambda} \right)_{\lambda=0}$.

Since the local frequency is very high, for large u , ϕ will propagate through the collapsing body and out to J^- by geometric optics. The null geodesic generators of the surfaces of constant phase that have a large u when they enter the body will have a v less than but infinitesimally close to $v = v_0 \equiv 0$ (where v_0 is the continuation backwards in time of the event horizon (which has $u = +\infty$)) as they pass the center (see fig. 5).

Introducing a geodesic deviation vector η^a between these generators and choosing its direction at J^- to be along $\left(\frac{\partial}{\partial v} \right)^a$, we realize that near $v = 0$, the v -dependence of ϕ at J^- will be the same as the dependence of $\phi(\lambda)$ on the affine parameter λ along the geodesic tangent to η^a , for points close to H^+ :

$$\phi(u(\lambda \leftarrow v)) \sim \begin{cases} 0 & , \quad v > 0 \\ e^{\frac{i\omega}{\kappa} \ln(-\alpha v)} & , \quad v < 0 . \end{cases} \quad (17)$$

The wave packet corresponding to $\phi(v)$ is

$$Z_{jnlm}(v) \equiv \frac{1}{\sqrt{E}} \int_{jE}^{(j+1)E} d\omega \left\{ \frac{1}{\sqrt{\omega}} \frac{\phi(v)}{r} Y_{l,m}(\theta, \phi) \right\} e^{\frac{2\pi i n \omega}{E}}$$

$$\sim \begin{cases} 0 & , \quad v > 0 \\ \frac{1}{L} e^{\frac{i\omega_j L}{E}} \sin\left(\frac{L}{2}\right) & , \quad v < 0, \end{cases}$$

where $\omega_j = \left(j + \frac{1}{2}\right) E$ is the original frequency of X_{jnlm} and $L = 2\pi n + \frac{E}{\kappa} \ln(-\alpha v)$. In the last line, we have also suppressed all v -independent factors.

The crucial point is that Z_{jnlm} is not purely positive frequency. In fact, it can be shown [4, Appendix A] that its Fourier transform $\hat{Z}_{jnlm}(\omega')$ satisfies

$$\hat{Z}_{jnlm}(-\omega') = -e^{-\frac{\pi\omega_j}{\kappa}} \hat{Z}_{jnlm}(\omega') \quad , \quad \omega' > 0. \quad (18)$$

So

$$\begin{aligned} Z_{jnlm}(v) &= Z_{jnlm}^{(+)}(v) + Z_{jnlm}^{(-)}(v) \\ &= \int_0^{+\infty} d\omega' \frac{e^{-i\omega'v}}{\sqrt{\omega'}} \hat{Z}_{jnlm}(\omega') \\ &+ \int_0^{+\infty} d\omega' \frac{e^{i\omega'v}}{\sqrt{\omega'}} \hat{Z}_{jnlm}(\omega') \left(-e^{-\frac{\pi\omega_j}{\kappa}}\right). \end{aligned}$$

We introduce the time reflected wave packet $\tilde{Z}_{jnlm}(v)$ at J^-

$$\begin{aligned} \tilde{Z}_{jnlm}(v) \equiv Z_{jnlm}(-v) &= \int_0^{+\infty} d\omega' \frac{e^{-i\omega'v}}{\sqrt{\omega'}} \hat{Z}_{jnlm}(\omega') \left(-e^{-\frac{\pi\omega_j}{\kappa}}\right) \\ &+ \int_0^{+\infty} d\omega' \frac{e^{i\omega'v}}{\sqrt{\omega'}} \hat{Z}_{jnlm}(\omega'). \end{aligned}$$

The $\tilde{Z}_{jnlm} : s$ are orthonormal with negative unit Klein-Gordon norm, and they are obviously orthogonal to the $\{Z_{jnlm}\}$. $\tilde{Z}_{jnlm}(v)$ will also propagate by geometric optics since its effective frequency is as high as $Z_{jnlm}(v)$: it will end up at the horizon just after its formation, see fig. 5. The resulting wave packet $J_{jnlm}(v)$ we choose to be part of our basis in $\overline{\mathcal{H}}_{H^+}$ and we denote them by ${}_i\overline{\tau}_a$. ${}_i\tau^a \in \mathcal{H}_{H^+}$ are thus early time horizon states.

We note that the combinations

$$\begin{aligned} & Z_{jnlm}(v) + e^{-\frac{\pi\omega_j}{\kappa}} \tilde{Z}_{jnlm}(v) \\ &= \int_0^{+\infty} d\omega' \frac{e^{-i\omega'v}}{\sqrt{\omega'}} \hat{Z}_{jnlm}(\omega') \left(1 - e^{-\frac{2\pi\omega_j}{\kappa}}\right), \end{aligned}$$

and

$$\begin{aligned} & Z_{jnlm}^*(v) + e^{\frac{\pi\omega_j}{\kappa}} \tilde{Z}_{jnlm}^*(v) \\ &= \int_0^{+\infty} d\omega' \frac{e^{-i\omega'v}}{\sqrt{\omega'}} \hat{Z}_{jnlm}^*(\omega') \left(e^{\frac{\pi\omega_j}{\kappa}} - e^{-\frac{\pi\omega_j}{\kappa}}\right), \end{aligned}$$

are purely positive frequency wave packets at J^- .

This implies that

$$\begin{cases} DC^{-1}{}_i \lambda^a &= e^{-\frac{\pi\omega_i}{\kappa}}{}_i \bar{\tau}_a \\ DC^{-1} \left(e^{\frac{\pi\omega_i}{\kappa}}{}_i \tau^a \right) &= {}_i \bar{\lambda}_a. \end{cases}$$

The action of $E = \overline{D}\overline{C}^{-1}$ on the basis elements ${}_i \bar{\gamma}_a, {}_i \bar{\lambda}_a, {}_i \bar{\tau}_a$ in $\overline{\mathcal{H}}_{H^+} \oplus \overline{\mathcal{H}}_{J^+}$ is therefore

$$\begin{cases} E {}_i \bar{\gamma}_a &= 0 \\ E {}_i \bar{\lambda}_a &= e^{-\frac{\pi\omega_i}{\kappa}}{}_i \tau^a \\ E {}_i \bar{\tau}_a &= e^{-\frac{\pi\omega_i}{\kappa}}{}_i \lambda^a. \end{cases}$$

Hence ε^{ab} is

$$\varepsilon^{ab} = \sum_i e^{-\frac{\pi\omega_i}{\kappa}} 2{}_i \lambda^{(a}{}_i \tau^{b)} + \varepsilon_0^{ab},$$

where ε_0^{ab} is orthogonal to all the late time basis vectors $\{{}_i \lambda^a\}$ and $\{{}_i \gamma^a\}$ as well as the early time horizon states $\{{}_i \tau^a\}$.

The state vector $|\psi\rangle = S|0_{in}\rangle$ corresponding to the in-vacuum was given by (16)

$$\psi = \psi(\varepsilon^{ab}) = \left(1, 0, \sqrt{\frac{1}{2}}\varepsilon^{ab}, 0, \sqrt{\frac{3.1}{4.2}}\varepsilon^{(ab}\varepsilon^{cd)}, 0, \dots\right).$$

Under the isomorphism between

$$\mathcal{F}_{\mathcal{I} \sqcup \mathcal{I}'} \left\{ \left(\oplus_{\mathcal{I}} \mathcal{H}_{\mathcal{I}} \right) \oplus \left(\oplus_{\mathcal{I}'} \mathcal{H}_{\mathcal{I}'} \right) \oplus \mathcal{H}(\varepsilon_i^{\mathcal{I} \sqcup \mathcal{I}'}) \right\},$$

and

$$\left\{ \otimes_i \mathcal{F}_{\cap \sqcup} (\mathcal{H}_\gamma) \right\} \otimes \left\{ \otimes_k \mathcal{F}_{\cap \sqcup} (\mathcal{H}_\parallel) \right\} \otimes \mathcal{F}_{\cap \sqcup} (\mathcal{H}(\varepsilon_i^{\perp \parallel})) ,$$

where \mathcal{H}_γ and \mathcal{H}_\parallel are the Hilbert spaces spanned by $\{i\lambda^a, {}_i\tau^a\}$ resp. $\{k\gamma^a\}$ and $\mathcal{H}(\varepsilon_i^{\perp \parallel})$ is the Hilbert space spanned by all other basis elements of $\mathcal{H}_{\mathcal{H}^+} \oplus \mathcal{H}_{\mathcal{J}^+}$, ψ is mapped to the following product state

$$\psi(\varepsilon^{ab}) = (\otimes_i \psi_i) \otimes (\otimes_k (\psi_0)_k) \otimes \psi(\varepsilon_0^{ab}) ,$$

where

$$\psi_i = \left(1, 0, \sqrt{\frac{1}{2}} e^{-\frac{\pi\omega_i}{\kappa}} {}_2i\lambda^{(a} {}_i\tau^{b)}, 0, \sqrt{\frac{3.1}{4.2}} e^{-2\frac{\pi\omega_i}{\kappa}} {}_4i\lambda^{(a} {}_i\tau^b {}_i\lambda^c {}_i\tau^{d)}, 0, \dots \right) . \quad (19)$$

$(\psi_0)_k$ is the vacuum state of $\mathcal{F}_{\cap \sqcup} (\mathcal{H}_\parallel)$ and

$$\psi = \psi(\varepsilon_0^{ab}) = \left(1, 0, \sqrt{\frac{1}{2}} \varepsilon_0^{ab}, 0, \sqrt{\frac{3.1}{4.2}} \varepsilon_0^{(ab} \varepsilon_0^{cd)}, 0, \dots \right) .$$

All information about the measurements on a given mode i is thus contained in the pure state $\psi_i \otimes (\psi_0)_i$ and there are no correlations between the different modes because the total state is the product of the states for the different modes.

To obtain the density matrix describing the measurements on a given mode i at J^+ , we must trace over the Fock space $\mathcal{F}_{\sigma_\gamma} (\mathcal{H}_{\sigma_\gamma}) \otimes \mathcal{F}_{\tau_\gamma} (\mathcal{H}_{\tau_\gamma})$ where $\mathcal{H}_{\sigma_\gamma}$ and $\mathcal{H}_{\tau_\gamma}$ are spanned by σ_i resp. τ_i .

First, we trace over $\mathcal{F}_{\tau_\gamma}$:

$$\begin{aligned} \tilde{D}_{\psi_i} &= \sum_{N, N', M} \langle \lambda_i^N, \tau_i^M | \psi_i \rangle \langle \psi_i | \lambda_i^{N'}, \tau_i^M \rangle | \lambda_i^N \rangle \langle \lambda_i^{N'} | \\ &= \sum_{N=0}^{+\infty} \left| \langle \lambda_i^N, \tau_i^N | \psi_i \rangle \right|^2 | \lambda_i^N \rangle \langle \lambda_i^N | . \end{aligned}$$

But $\left| \langle \lambda_i^N, \tau_i^N | \psi_i \rangle \right|^2$, which is proportional to the probability \tilde{P}_N for observing N "particles" in state λ_i , is obtained by taking the squared norm of the vector in ψ_i which is proportional to $|\lambda_i^N, \tau_i^N\rangle$

$$\begin{aligned} \tilde{P}_N &\propto \frac{(2N-1)(2N-3)\dots 1}{2N(2N-2)\dots 2} e^{-2N\frac{\pi\omega_i}{\kappa}} 2^{2N} \left\| {}_i\lambda^{(a_1} {}_i\tau^{a_2} \dots {}_i\lambda^{a_{2N-1}} {}_i\tau^{a_{2N}}) \right\|^2 \\ &= \frac{(2N-1)(2N-3)\dots 1}{2N(2N-2)\dots 2} e^{-2N\frac{\pi\omega_i}{\kappa}} 2^{2N} \frac{N!N!}{(2N)!} \\ &= e^{-N\frac{2\pi\omega_i}{\kappa}} . \end{aligned} \quad (20)$$

This is precisely the Boltzman factor with a temperature T given by

$$kT = \frac{\hbar\kappa}{2\pi} . \quad (21)$$

The density matrix obtained by tracing over ${}_i\tau^a$ is therefore

$$\tilde{D}_{\psi_i} = \sum_{n=0}^{+\infty} e^{-n \frac{\hbar\omega_i}{kT}} |\lambda_i^n\rangle \langle \lambda_i^n| .$$

For each state $|\lambda_i^n\rangle$, we obtain a density matrix obtained by tracing over \mathcal{F}_{σ_i} :

$$\begin{aligned} D_{\lambda_i^n} &= \sum_{N,N',M} \langle \rho_i^N, \sigma_i^M | \lambda_i^n \rangle \langle \lambda_i^n | \rho_i^{N'}, \sigma_i^M \rangle |\rho_i^N\rangle \langle \rho_i^{N'}| \\ &= \sum_{N=0}^n \left| \langle \rho_i^N, \sigma_i^{n-N} | \lambda_i^n \rangle \right|^2 |\rho_i^N\rangle \langle \rho_i^N| , \end{aligned}$$

since $\lambda_i^n = (t_i \rho_i + r_i \sigma_i)^n = \sum_{N=0}^n \binom{n}{N} t_i^N r_i^{n-N} \rho_i^{(N)} \sigma_i^{n-N}$.

Setting

$$\Gamma_i = |t_i|^2 \Rightarrow 1 - \Gamma_i = |r_i|^2 ,$$

and using

$$\left\| \binom{n}{N} \rho_i^{(N)} \sigma_i^{n-N} \right\|^2 = \binom{n}{N} ,$$

we obtain

$$D_{\lambda_i^n} = \sum_{N=0}^n \binom{n}{N} \Gamma_i^N (1 - \Gamma_i)^{n-N} |\rho_i^N\rangle \langle \rho_i^N| .$$

Defining $x = e^{-\frac{\hbar\omega_i}{kT}}$, the density matrix describing the measurements on a given

mode i at J^+ becomes

$$\begin{aligned}
D_{\psi_i} &= \sum_{n=0}^{+\infty} x^n D_{\lambda_i^n} = \sum_{n=0}^{+\infty} x^n \sum_{N=0}^n \binom{n}{N} \Gamma_i^N (1 - \Gamma_i)^{n-N} |\rho_i^N\rangle \langle \rho_i^N| \\
&= \sum_{N=0}^{+\infty} |\rho_i^N\rangle \langle \rho_i^N| \sum_{n=N}^{+\infty} \binom{n}{N} \Gamma_i^N (1 - \Gamma_i)^{n-N} x^n \\
&= \sum_{N=0}^{+\infty} |\rho_i^N\rangle \langle \rho_i^N| \left\{ (\Gamma_i x)^N \sum_{n'=0}^{+\infty} \binom{n'+N}{N} ((1 - \Gamma_i) x)^{n'} \right\} \\
&= \sum_{N=0}^{+\infty} |\rho_i^N\rangle \langle \rho_i^N| \frac{(\Gamma_i x)^N}{(1 - (1 - \Gamma_i) x)^{N+1}} .
\end{aligned}$$

Multiplying D_{ψ_i} by $(1 - (1 - \Gamma_i) x)$, we finally get

$$D_{\psi_i} = \sum_{N=0}^{+\infty} \left(\frac{\Gamma_i e^{-\frac{\hbar\omega_i}{kT}}}{1 - (1 - \Gamma_i) e^{-\frac{\hbar\omega_i}{kT}}} \right)^N |\rho_i^N\rangle \langle \rho_i^N| . \quad (22)$$

Writing

$$x' = \frac{\Gamma_i e^{-\frac{\hbar\omega_i}{kT}}}{1 - (1 - \Gamma_i) e^{-\frac{\hbar\omega_i}{kT}}} ,$$

we see that the probability for observing N particles in the mode i at J^+ is

$$P_N = \frac{x'^N}{\sum_{M=0}^{+\infty} x'^M} = (1 - x') x'^N ,$$

and the average number of particles is

$$\begin{aligned}
\sum_{N=0}^{+\infty} N (1 - x') x'^N &= x' \frac{\partial}{\partial x'} \ln \frac{1}{1 - x'} = \frac{x'}{1 - x'} \\
&= \Gamma_i \frac{e^{-\frac{\hbar\omega_i}{kT}}}{1 - e^{-\frac{\hbar\omega_i}{kT}}} .
\end{aligned} \quad (23)$$

The most important result of this section is that the black hole emits particles with a *thermal spectrum* at temperature $kT = \hbar\kappa/2\pi$. As a consequence, the quantum state describing these particles is mixed, that is, it is described by a density matrix. If the black hole evaporates completely, and Hawking's semi-classical calculation is assumed to be valid throughout the evaporation process,

then, starting from the pure state $|0_{in}\rangle$ at J^- , it will evolve to the mixed thermal state at J^+ [5]. The relation between incoming and outgoing state is therefore not given by a unitary S -matrix, and quantum mechanics appear to be violated: this is the essence of Hawking's paradox.

4 Two Dimensional Black Holes

4.1 Classical Two Dimensional Black Holes

The main problem in the study of black hole evaporation in four dimensions is to include the back reaction of the matter fields on the background geometry of the black hole: one would like to solve the semiclassical Einstein equations

$$G_{ab} = 8\pi \langle \psi | \hat{T}_{ab} | \psi \rangle , \quad (24)$$

where $\langle \psi | \hat{T}_{ab} | \psi \rangle$ is the expectation value of the stress-energy tensor \hat{T}_{ab} of the matter fields in a state $|\psi\rangle$.

$\langle \psi | \hat{T}_{ab} | \psi \rangle$ is divergent but it can be regularized. The divergent terms, which do not depend on $|\psi\rangle$, can be obtained from an effective action containing two terms that are quadratic in the curvature tensor, see [6, chapter 6].

Thus, the renormalized expectation value $\langle \psi | \hat{T}_{ab} | \psi \rangle_{ren}$ has a two parameter ambiguity. These two terms are fourth order (i.e., they contain four derivatives) and lead to instabilities in the semiclassical equations. Besides that, $\langle \psi | \hat{T}_{ab} | \psi \rangle_{ren}$ contains terms which are nonlocal so that in four dimensions, the problem seems intractable.

On the other hand, in two dimensions, the renormalized expectation value of the stress-energy tensor contains two derivatives of the metric, so the semiclassical Einstein equations remain second order.

In two dimensions, the Einstein-Hilbert action

$$S = \frac{1}{4\pi} \int d^2x \sqrt{-g} R ,$$

is a topological invariant because

$$\delta S = \frac{1}{4\pi} \int d^2x \sqrt{-g} \left\{ \left(R_{ab} - \frac{1}{2} g_{ab} R \right) \delta g^{ab} + g_{ab} \nabla^2 \delta g^{ab} - \nabla_a \nabla_b \delta g^{ab} \right\} ,$$

where we have used

$$\begin{cases} g^{ab} \delta R_{ab} &= g_{ab} \nabla^2 \delta g^{ab} - \nabla_a \nabla_b \delta g^{ab} \\ \delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab} . \end{cases}$$

In two dimensions

$$R_{ab} \equiv \frac{1}{2} g_{ab} R ,$$

for all metrics. Therefore, δS is the integral of a total divergence and we have no classical equations of motion. It can be shown that for a compact two dimensional manifold of genus g , the Einstein action gives the so-called Euler characteristic

$$\frac{1}{4\pi} \int d^2x \sqrt{g} R = \chi = 2(1 - g) .$$

To get a dynamical theory of gravity in two dimensions, one can couple gravity to the scalar dilaton field Φ . This coupling is realized in string theory, where Φ is one of the massless modes of the closed string together with the graviton and an antisymmetric tensor field. In the low energy limit, the Lagrangian describing this interaction is, in four dimensions, [7, chapter 3]

$$S = \int d^4x \sqrt{-g} e^{-2\Phi} \left\{ R + 4(\nabla\Phi)^2 - \frac{1}{2} g^{ae} g^{bf} F_{ab} F_{ef} \right\} ,$$

where we have added a Maxwell field associated with a $U(1)$ subgroup of $E_8 \otimes E_8$ or $spin(32)/Z_2$ and we have set to zero the remaining gauge fields and antisymmetric tensor field.

Defining a new metric \tilde{g} , with $\tilde{g} = e^{-2\Phi} g$, the action becomes

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \tilde{R} - 2\tilde{g}^{ab} \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} e^{-2\Phi} \tilde{g}^{ae} \tilde{g}^{bf} F_{ab} F_{ef} \right\} .$$

Here we have used that

$$\begin{cases} \sqrt{-g} &= e^{4\Phi} \sqrt{-\tilde{g}} \\ R &= e^{-2\Phi} \left(\tilde{R} - 6 \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \Phi - 6 \tilde{g}^{ab} \tilde{\nabla}_a \Phi \tilde{\nabla}_b \Phi \right) , \end{cases}$$

if $g = e^{+2\Phi} \tilde{g}$ in four dimensions.

When $F_{ab} = 0$, this action describes ordinary Einstein gravity coupled to a massless Klein-Gordon field. The no-hair theorems then imply that the unique stationary black hole solutions are described by the Kerr metric (1) with zero charge.

When $F_{ab} \neq 0$, $\Phi \neq 0$, and the solution is not of the Kerr form: the metric, dilaton and electromagnetic field tensor for a non-rotating, magnetically charged and spherically symmetric black hole are [8]:

$$\begin{cases} d\tilde{s}^2 &= \tilde{g}_{ab} dx^a dx^b \\ &= - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r \left(r - \frac{Q^2}{2M} e^{-2\Phi_0} \right) d\Omega^2 \\ e^{-2\Phi} &= e^{-2\Phi_0} \left(1 - \frac{Q^2}{2Mr} e^{-2\Phi_0} \right) \\ F_{ab} &= Q \sin \theta \, 2 d\theta_{[a} d\phi_{b]} . \end{cases}$$

Here, Φ_0 is the value of the dilaton at infinity.

This metric is almost the same as the Schwarzschild metric. The only difference

is that the areas of the two-spheres are decreased relative to the Schwarzschild values

$$\frac{A}{4\pi} = r^2 \left(1 - \frac{Q^2}{2Mr} e^{-2\Phi_0} \right) = r^2 e^{-2(\Phi - \Phi_0)} .$$

The area is zero when $r = \frac{Q^2}{2M} e^{-2\Phi_0}$, causing this surface to be singular, but the horizon is still located at $r = 2M$.

When $\frac{Q^2}{2M} e^{-2\Phi_0} < 2M$, the singularity is inside the horizon; when $\frac{Q^2}{2M} e^{-2\Phi_0} = 2M$, the horizon becomes singular and when $\frac{Q^2}{2M} e^{-2\Phi_0} > 2M$ the solution has a naked singularity.

Comparing with the charged solutions of the Einstein-Maxwell theory, given by the Reissner-Nordstrom metric (4), we note several differences:

First, when the dilaton is present, there is no inner horizon.

Second, the transition from black hole to naked singularity occurs at $\frac{Q^2}{2M} e^{-2\Phi_0} = 2M$ rather than $Q^2 = M^2$ as for Reissner-Nordstrom black holes and furthermore, at this transition (the black hole is then called an extremal hole), the horizon is singular in the first case, but completely regular in the second case.

However, for extremal dilaton black holes, the metric seen by the string is $g = e^{2\Phi} \tilde{g}$, i.e.,

$$ds^2 = g_{ab} dx^a dx^b = e^{2\Phi_0} \left(-dt^2 + \left(1 - \frac{2M}{r} \right)^{-2} dr^2 + r^2 d\Omega^2 \right) .$$

Defining the coordinates τ, σ by

$$\begin{cases} \tau &= e^{\Phi_0} t \\ d\sigma &= e^{\Phi_0} \left(1 - \frac{2M}{r} \right)^{-1} dr \Rightarrow \sigma = e^{\Phi_0} \left(r + 2M \ln \left(\frac{r}{2M} - 1 \right) \right) , \end{cases}$$

we get

$$\begin{cases} ds^2 &= -d\tau^2 + d\sigma^2 + e^{2\Phi_0} r^2 d\Omega^2 \\ e^{-2\Phi} &= e^{-2\Phi_0} \left(1 - \frac{2M}{r} \right) = e^{-2\Phi_0} \frac{2M}{r} \exp \left(\frac{e^{-\Phi_0} \sigma - r}{2M} \right) . \end{cases}$$

When $r \rightarrow +\infty$, the geometry approaches that of flat four dimensional space; when $r \rightarrow 2M$ or $\sigma \rightarrow -\infty$, it approaches that of flat two dimensional space in the variable (σ, τ) times a two sphere of radius $R = 2M e^{\Phi_0} = |Q|$ and

$$-2\Phi \sim \frac{\sigma}{2M e^{\Phi_0}} \Leftrightarrow \Phi \sim -\lambda \sigma ,$$

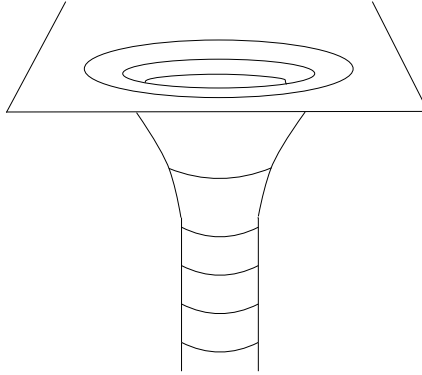


Figure 6:

where $\lambda = \frac{1}{2Q}$.

Thus, the geometry seen by the string is free of horizons and singularities in contrast to extremal Reissner-Nordstrom black holes. This geometry is that of a bottomless hole, see fig. 6. In the throat region, where $r \rightarrow 2M$, the dilaton field is linear in σ . The extremal dilaton black hole is consequently called the linear dilaton vacuum. It is called vacuum because it is stable, i.e., it does not emit any Hawking radiation. This is because the string metric is free of horizons and singularities and thus, the Hawking temperature is zero, as for extremal Reissner-Nordstrom black holes (see (9)).

For non-extremal black holes, the string metric still describes a black hole with an event horizon and a singularity because the conformal factor $e^{2\Phi}$ is finite everywhere outside (and on) the horizon.

At low energies compared to $\frac{1}{R}$, the physics in the throat region of an extremal or near extremal black hole can be described by an effective two dimensional action

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\Phi} \left\{ R + 4(\nabla\Phi)^2 + 4\lambda^2 - \frac{1}{2}F^2 \right\} .$$

Because the electromagnetic field tensor F has no propagating degrees of freedom in two dimensions, we can set it to zero if no charged particles are present.

We will see later that this two dimensional model indeed has a linear dilaton vacuum solution and also that a particle thrown into this dilaton vacuum turns it into a non-extremal black hole with a horizon and a singularity.

We note that in the throat region of a near extremal black hole, the area of

two-spheres, as measured in the canonical metric \tilde{g} , is

$$\frac{A}{4\pi} = r^2 e^{-2(\Phi-\Phi_0)} \approx 4M^2 e^{2\Phi_0} e^{-2\Phi} \approx Q^2 e^{-2\Phi} .$$

The area of two-spheres in the four dimensional theory is thus proportional to $e^{-2\Phi}$. This fact will be useful later when we will study the two dimensional theory without referring to its four dimensional origin.

Thus, we will start from the action

$$\frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\Phi} \left\{ R + 4(\nabla\Phi)^2 + 4\lambda^2 \right\} , \quad (25)$$

and look at it as a model of quantum gravity in two dimensions. This model was proposed in 1991 by Callan, Giddings, Harvey and Strominger [9] and is therefore usually referred to as the CGHS model.

Another model of two dimensional gravity is the one obtained from the four dimensional Einstein-Hilbert action by restricting oneself to spherical symmetric metrics (For the definition of x^\pm , see (28))

$$\begin{aligned} S_4 &\propto \int d^4x \sqrt{-g_4} R_4 , \\ ds^2 &= (g_4)_{ab} dx^a dx^b = -e^{2\rho} dx^+ dx^- + e^{-2\Phi} d\Omega^2 , \end{aligned}$$

or

$$((g_4)_{ab}) = \begin{pmatrix} 0 & -\frac{1}{2}e^{2\rho} & 0 & 0 \\ -\frac{1}{2}e^{2\rho} & 0 & 0 & 0 \\ 0 & 0 & e^{-2\Phi} & 0 \\ 0 & 0 & 0 & e^{-2\Phi} \sin^2 \theta \end{pmatrix} .$$

This implies

$$\sqrt{-g_4} = \sqrt{-g_2} e^{-2\Phi} \sin \theta .$$

It can also be shown that (see [2, exercise 14.16])

$$\begin{aligned} R_4 &= 2e^{2\Phi} + e^{-2\rho} (8\partial_+ \partial_- \rho + 24\partial_+ \Phi \partial_- \Phi - 16\partial_+ \partial_- \Phi) \\ &= 2e^{2\Phi} + R_2 - 6(\nabla\Phi)^2 + 4\nabla^2\Phi , \end{aligned}$$

where we have used that $R_2 = 8e^{-2\rho} \partial_+ \partial_- \rho$ (see (29)).

Therefore

$$S_2 \propto \int d^2x \sqrt{-g_2} e^{-2\Phi} \left\{ R_2 + 2(\nabla\Phi)^2 + 2e^{2\Phi} \right\} ,$$

where we have integrated by parts once. This action looks similar to the CGHS action with the same interpretation for $e^{-2\Phi}$.

The CGHS action is

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left\{ e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 + 4\lambda^2 \right) - \frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right\}, \quad (26)$$

where we have added N scalar massless matter fields.

To obtain the classical equations of motion for the metric g , we vary S with respect to g . We use the following relations

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab} \\ \delta R &= \delta(R_{ab} g^{ab}) = \delta R_{ab} g^{ab} + R_{ab} \delta g^{ab}, \end{aligned}$$

where

$$g^{ab} \delta R_{ab} = g_{ab} \nabla^2 \delta g^{ab} - \nabla_a \nabla_b \delta g^{ab},$$

and

$$R_{ab} = \frac{1}{2} g_{ab} R.$$

Then

$$\begin{aligned} \delta_g S &= \frac{1}{2\pi} \int d^2x \sqrt{-g} \left\{ -\frac{1}{2} \left(e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f_i)^2 \right) g_{ab} \right. \\ &+ \left(e^{-2\Phi} 4 \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} \nabla_a f_i \nabla_b f_i \right) \\ &+ \left. \left(e^{-2\Phi} \frac{1}{2} g_{ab} R + g_{ab} \nabla^2 e^{-2\Phi} - \nabla_a \nabla_b e^{-2\Phi} \right) \right\} \delta g^{ab} \\ &= \frac{1}{2\pi} \int d^2x \sqrt{-g} \left\{ -\frac{1}{2} \left(e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f_i)^2 \right) g_{ab} \right. \\ &+ \left(e^{-2\Phi} 4 \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} \nabla_a f_i \nabla_b f_i \right) \\ &+ \left. e^{-2\Phi} \left(\frac{1}{2} g_{ab} R + \left(4(\nabla\Phi)^2 - 2\nabla^2\Phi \right) g_{ab} - (4\nabla_a \Phi \nabla_b \Phi - 2\nabla_a \nabla_b \Phi) \right) \right\} \delta g^{ab}. \end{aligned}$$

Setting the variation to zero, we obtain the equations of motion

$$\begin{aligned} &e^{-2\Phi} \left(2\nabla_a \nabla_b \Phi - 2g_{ab} \left(\nabla^2\Phi - (\nabla\Phi)^2 + \lambda^2 \right) \right) \\ &= \frac{1}{2} \left(\nabla_a f_i \nabla_b f_i - \frac{1}{2} (\nabla f_i)^2 g_{ab} \right) \equiv T_{ab}^f. \end{aligned}$$

These equations correspond to the Einstein equations in ordinary four dimensional gravity.

Varying the action with respect to Φ

$$\begin{aligned}
\delta_\Phi S &= \frac{1}{2\pi} \int d^2x \sqrt{-g} \left\{ \left(R + 4(\nabla\Phi)^2 + 4\lambda^2 \right) \delta e^{-2\Phi} + e^{-2\Phi} 4\delta(\nabla\Phi)^2 \right\} \\
&= \frac{1}{2\pi} \int d^2x \sqrt{-g} \left\{ -2e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 + 4\lambda^2 \right) - 8\nabla \cdot \left(e^{-2\Phi} \nabla\Phi \right) \right\} \delta\Phi \\
&= \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\Phi} \left\{ -2 \left(R + 4(\nabla\Phi)^2 + 4\lambda^2 \right) \right. \\
&\quad \left. - 8 \left(\nabla^2\Phi - 2(\nabla\Phi)^2 \right) \right\} \delta\Phi = 0 ,
\end{aligned}$$

one gets

$$\frac{R}{4} + \nabla^2\Phi - (\nabla\Phi)^2 + \lambda^2 = 0 .$$

Finally

$$\delta_f S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \nabla^2 f_i \delta f_i = 0 ,$$

or

$$\nabla^2 f_i = 0 .$$

The equations of motion are thus

$$\left\{ \begin{array}{ll} 2e^{-2\Phi} \left(\nabla_a \nabla_b \Phi - g_{ab} \left(\nabla^2\Phi - (\nabla\Phi)^2 + \lambda^2 \right) \right) &= T_{ab}^f \\ \frac{R}{4} + \nabla^2\Phi - (\nabla\Phi)^2 + \lambda^2 &= 0 \\ \nabla^2 f_i &= 0 . \end{array} \right. \quad (27)$$

Since a reparametrization depends on two arbitrary functions of the old coordinates we can always find a coordinate system in which the two diagonal elements of g are vanishing. The new coordinates are therefore null coordinates and we call them x^\pm . Thus

$$\begin{aligned}
g_{++} &= g_{--} = 0 , \\
g_{+-} &= -\frac{1}{2} e^{2\rho} ,
\end{aligned} \quad (28)$$

or

$$ds^2 = -e^{2\rho} dx^+ dx^- .$$

This is called the conformal gauge.

In this gauge, the non-vanishing connections are

$$\Gamma_{++}^+ = 2 \partial_+ \rho, \quad \Gamma_{--}^- = 2 \partial_- \rho,$$

which gives

$$[\nabla_+, \nabla_-] \omega_+ = [\partial_+ - 2 \partial_+ \rho, \partial_-] \omega_+ = 2 \partial_+ \partial_- \rho \omega_+.$$

By definition

$$[\nabla_+, \nabla_-] \omega_+ = R_{+-+}^+ \omega_+,$$

that is

$$R_{+-+}^+ = 2 \partial_+ \partial_- \rho,$$

and

$$R_{++}^+ = R_{++}^{++} + R_{+-}^{+-} = -g^{+-} R_{+-+}^+ = 4e^{-2\rho} \partial_+ \partial_- \rho,$$

$$R_{--}^- = R_{++}^+.$$

Therefore, the curvature scalar R is

$$R = R_{++}^+ + R_{--}^- = 8 e^{-2\rho} \partial_+ \partial_- \rho. \quad (29)$$

In the conformal gauge, the equations of motion are:

$$g_{+-} : 2e^{-2\Phi} (\partial_+ \partial_- \Phi - g_{+-} (2g^{+-} \partial_+ \partial_- \Phi - 2g^{+-} \partial_+ \Phi \partial_- \Phi + \lambda^2))$$

$$= \frac{1}{2} \left(\partial_+ f_i \partial_- f_i - \frac{1}{2} g_{+-} 2g^{+-} \partial_+ f_i \partial_- f_i \right)$$

$$\Leftrightarrow 4\partial_+ \Phi \partial_- \Phi - 2\partial_+ \partial_- \Phi + \lambda^2 e^{2\rho} = 0$$

$$\Phi : 2e^{-2\rho} \partial_+ \partial_- \rho - 4e^{-2\rho} \partial_+ \partial_- \Phi + 4e^{-2\rho} \partial_+ \Phi \partial_- \Phi + \lambda^2 = 0$$

$$\Leftrightarrow 2\partial_+ \partial_- \rho - 4\partial_+ \partial_- \Phi + 4\partial_+ \Phi \partial_- \Phi + \lambda^2 e^{2\rho} = 0$$

$$f_i : \partial_+ \partial_- f_i = 0.$$

The remaining equations for g_{++} and g_{--} are constraints coming from the gauge fixing $g_{+-} = g_{-+} = 0$

$$\begin{aligned} g_{\pm\pm} &: 2e^{-2\Phi}\nabla_{\pm}\nabla_{\pm}\Phi = \frac{1}{2}\partial_{\pm}f_i\partial_{\pm}f_i \\ \Leftrightarrow e^{-2\Phi}\left(2\partial_{\pm}^2\Phi - 4\partial_{\pm}\rho\partial_{\pm}\Phi\right) &= \frac{1}{2}\partial_{\pm}f_i\partial_{\pm}f_i . \end{aligned}$$

So

$$\left\{ \begin{array}{lll} 4\partial_+\Phi\partial_-\Phi - 2\partial_+\partial_-\Phi + \lambda^2 e^{2\rho} & = & 0 \quad (\rho) \\ 2\partial_+\partial_-\rho - 4\partial_+\partial_-\Phi + 4\partial_+\Phi\partial_-\Phi + \lambda^2 e^{2\rho} & = & 0 \quad (\Phi) \\ \partial_+\partial_-\Phi & = & 0 \quad (f_i) \\ e^{-2\Phi}\left(2\partial_{\pm}^2\Phi - 4\partial_{\pm}\rho\partial_{\pm}\Phi\right) & = & \frac{1}{2}\partial_{\pm}f_i\partial_{\pm}f_i \quad (g_{\pm\pm}) . \end{array} \right. \quad (30)$$

Subtracting (ρ) from (Φ) we get

$$\Leftrightarrow \begin{aligned} \partial_+\partial_-(\rho - \Phi) &= 0 \\ \rho - \Phi &= f_+(x^+) + f_-(x^-) . \end{aligned}$$

The conformal gauge is preserved by arbitrary coordinate transformations of the form

$$\left\{ \begin{array}{l} x^+ \rightarrow \tilde{x}^+(x^+) \\ x^- \rightarrow \tilde{x}^-(x^-) \end{array} \right. ,$$

since

$$0 = g_{\pm\pm}(x) = \left(\frac{d\tilde{x}^{\pm}}{dx^{\pm}}\right)^2 \tilde{g}_{\pm\pm}(\tilde{x}) .$$

This fact allows us to choose coordinates such that $f_+(x^+) = f_-(x^-) = 0$, so we can set $\Phi = \rho$. For reasons that will be clear in the following, these coordinates will be called the Kruskal coordinates.

The equations of motion and constraints become:

$$\left\{ \begin{array}{lll} -\partial_+\partial_-\rho & = & \lambda^2 \quad (\rho) \\ \partial_+\partial_-\Phi & = & 0 \quad (f_i) \\ -\partial_{\pm}^2\rho & = & T_{\pm\pm}^f \left(= \frac{1}{2}\partial_{\pm}f_i\partial_{\pm}f_i\right) \quad (g_{\pm\pm}) . \end{array} \right. \quad (31)$$

The general static solution when $f_i = 0$ is

$$e^{-2\Phi} = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2 x^+ x^- .$$

In the following, M is no longer the mass of the four dimensional black hole although we use the same letter. In fact, this new M turns out to be the mass of the *two dimensional* black hole.

When $M = 0$

$$ds^2 = -e^{2\rho} dx^+ dx^- = -\left(-\lambda^2 x^+ x^-\right)^{-1} dx^+ dx^- .$$

Defining the coordinates σ^\pm by

$$\lambda x^\pm = \pm e^{\pm\lambda\sigma^\pm} \Rightarrow dx^\pm = \pm \lambda x^\pm d\sigma^\pm ,$$

we see that the spacetime is flat:

$$ds^2 = -d\sigma^+ d\sigma^- = -d\tau^2 + d\sigma^2 ,$$

where

$$\sigma^\pm = \tau \pm \sigma .$$

Furthermore

$$e^{-2\Phi} = -\lambda^2 x^+ x^- = e^{\lambda(\sigma^+ - \sigma^-)} = e^{2\lambda\sigma} ,$$

or

$$\Phi = -\lambda\sigma .$$

We thus recognize the linear dilaton vacuum described earlier. In the four dimensional theory, this solution corresponds to an extremal dilaton black hole solution.

When $M \neq 0$, the scalar curvature is

$$\begin{aligned} R &= 8e^{-2\rho} \partial_+ \partial_- \rho = 4 \left(e^{2\rho} \partial_+ e^{-2\rho} \partial_- e^{-2\rho} - \partial_+ \partial_- e^{-2\rho} \right) \\ &= 4 \left(\left(\frac{M}{\lambda} - \lambda^2 x^+ x^- \right)^{-1} \left(-\lambda^2 x^- \right) \left(-\lambda^2 x^+ \right) + \lambda^2 \right) \\ &= \frac{4M\lambda}{\frac{M}{\lambda} - \lambda^2 x^+ x^-} . \end{aligned} \tag{32}$$

At $\lambda^3 x^+ x^- = M$, we have a curvature singularity asymptotically approaching the

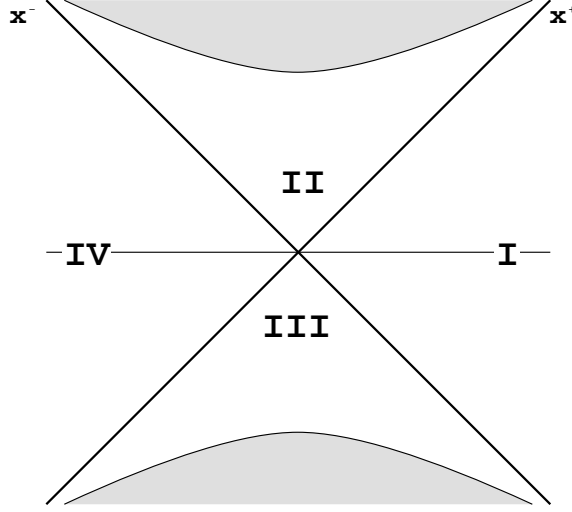


Figure 7:

null curves $x^\pm = 0$, which therefore are event horizons. The spacetime diagram of this black hole solution, depicted in fig. 7, is qualitatively similar to that of the extended Schwarzschild solution reported in fig. 1.

In the four dimensional theory, these black hole solutions correspond to near extremal dilaton black holes.

Using the coordinates σ^\pm introduced earlier, we see that

$$\begin{aligned}
ds^2 &= -\left(\frac{M}{\lambda} - \lambda^2 x^+ x^-\right)^{-1} dx^+ dx^- \\
&= -\left(\frac{M}{\lambda} + e^{\lambda(\sigma^+ - \sigma^-)}\right)^{-1} e^{\lambda(\sigma^+ - \sigma^-)} d\sigma^+ d\sigma^- \\
&= -\left(\frac{M}{\lambda} e^{-2\lambda\sigma} + 1\right)^{-1} d\sigma^+ d\sigma^- = -e^{2\rho'} d\sigma^+ d\sigma^-.
\end{aligned}$$

Since the components of the metric do not depend on τ , $\left(\frac{\partial}{\partial\tau}\right)^a$ is the timelike Killing field of the manifold. Therefore, τ corresponds to t in the Schwarzschild metric.

We also see that region I of fig. 7 is flat at spatial and null infinity J_R^\pm : as $\sigma \rightarrow +\infty$

$$2\rho' \sim -\frac{M}{\lambda} e^{-2\lambda\sigma}$$

which vanishes exponentially.

σ corresponds to r^* in the Schwarzschild metric. Thus, σ^\pm correspond to $\left\{ \begin{matrix} v \\ u \end{matrix} \right\}$

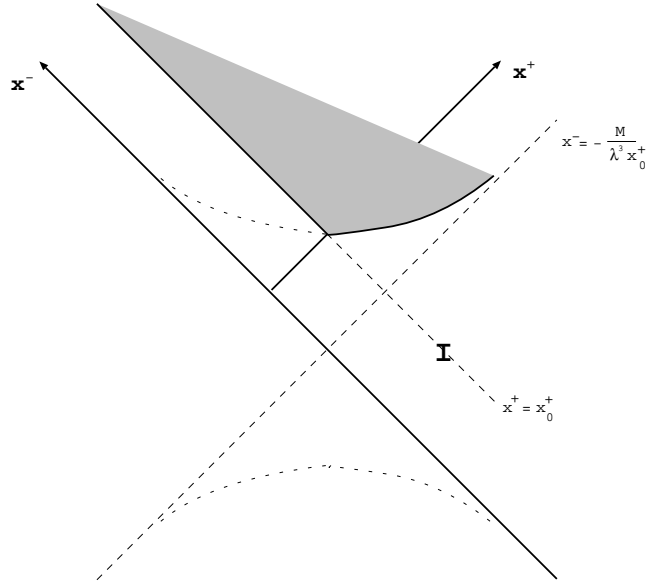


Figure 8:

and x^\pm correspond to $\begin{Bmatrix} V \\ U \end{Bmatrix}$.

Also, when $\sigma \rightarrow +\infty$

$$\begin{aligned}
e^{-2\Phi} &= \frac{M}{\lambda} - \lambda^2 x^+ x^- = \frac{M}{\lambda} + e^{2\lambda\sigma} \\
\Rightarrow e^{-2(\Phi+\lambda\sigma)} &= 1 + \frac{M}{\lambda} e^{-2\lambda\sigma} \\
\Rightarrow \Phi &\sim -\lambda\sigma - \frac{M}{2\lambda} e^{-2\lambda\sigma} .
\end{aligned}$$

Therefore, we have shown that region I of the two dimensional black hole asymptotically approaches the linear dilaton vacuum at null infinity.

We can patch together the vacuum solution with the black hole solution across some light-like line $x^+ = x_0^+$ as follows

$$e^{-2\Phi} = e^{-2\rho} = \begin{cases} -\lambda^2 x^+ x^- & , \quad x^+ < x_0^+ \\ \frac{M}{\lambda} - \lambda^2 x^+ \left(x^- + \frac{M}{\lambda^3 x_0^+} \right) & , \quad x^+ > x_0^+ . \end{cases} \quad (33)$$

The corresponding spacetime diagram and Penrose diagram are shown in fig. 8 resp. fig. 9.

For $x^+ > x_0^+$, the solution is identical to the black hole solution but it is translated by $\Delta x^- = -\frac{M}{\lambda^3 x_0^+}$. In particular, the event horizon is located at $x^- = -\frac{M}{\lambda^3 x_0^+}$.

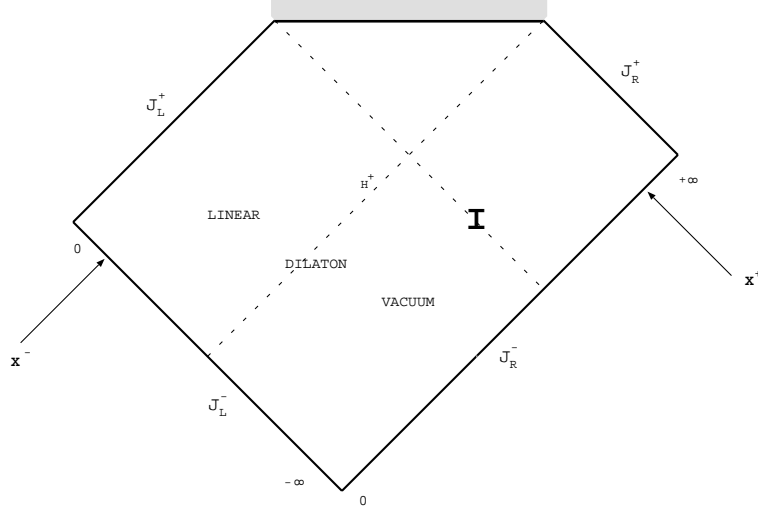


Figure 9:

There is a discontinuity in $\partial_+ e^{-2\rho}$:

$$\begin{aligned} \partial_+ e^{-2\rho} &= \begin{cases} -\lambda^2 x^- & , \quad x^+ < x_0^+ \\ -\lambda^2 \left(x^- + \frac{M}{\lambda^3 x_0^+} \right) & , \quad x^+ > x_0^+ \end{cases} \\ \Rightarrow -\partial_+^2 e^{-2\rho} &= \frac{M}{\lambda x_0^+} \delta(x^+ - x_0^+) . \end{aligned}$$

The constraint (g_{++}) is thus satisfied for

$$T_{++}^f = \frac{1}{2} \partial_+ f \partial_+ f = \frac{M}{\lambda x_0^+} \delta(x^+ - x_0^+) ,$$

where we have specialized to a single massless scalar field f .

The asymptotically minkowskian coordinates σ^\pm are now given by

$$\begin{cases} \lambda x^+ & = e^{\lambda \sigma^+} \\ \lambda \left(x^- + \frac{M}{\lambda^3 x_0^+} \right) & = -e^{-\lambda \sigma^-} , \end{cases}$$

and they cover region I of fig. 9.

The energy carried by the incoming field f is most easily defined and calculated in the asymptotically flat region J_R^- . We use the asymptotically minkowskian coordinates (τ, σ) because energy is defined as the component of energy-momentum along minkowskian time, that is, τ .

The energy-momentum flow through the volume one-form

$$\Sigma_a = \varepsilon_{ab} \left(\frac{\partial}{\partial \sigma^+} \right)^b = \varepsilon_{\sigma^- \sigma^+} (d\sigma^-)_a = \sqrt{-g} (d\sigma^-)_a = \frac{1}{2} (d\sigma^-)_a ,$$

is, by definition [2, chapter 5]

$$T^{ab}\Sigma_b = T^{\sigma^-\sigma^-} \left(\frac{\partial}{\partial\sigma^-} \right)^a \left(\frac{\partial}{\partial\sigma^-} \right)^b \frac{1}{2} (d\sigma^-)_b = \frac{1}{2} T^{\sigma^-\sigma^-} \left(\frac{\partial}{\partial\sigma^-} \right)^a .$$

Since $\sigma^\pm = \tau \pm \sigma$ or $(\partial/\partial\sigma^\pm)^a = \frac{1}{2}((\partial/\partial\tau)^a \pm (\partial/\partial\sigma)^a)$, the energy flow in the σ^- -direction per unit length in the σ^+ - direction is

$$\frac{1}{4} T^{\sigma^-\sigma^-} = (g_{\sigma^+\sigma^-})^2 T^{\sigma^-\sigma^-} = T_{\sigma^+\sigma^+} .$$

The total energy carried by f is

$$\begin{aligned} \int_{-\infty}^{+\infty} d\sigma^+ T_{\sigma^+\sigma^+}^f &= \int_{-\infty}^{+\infty} dx^+ \left(\frac{d\sigma^+}{dx^+} \right) \left(\frac{dx^+}{d\sigma^+} \right)^2 T_{++}^f \\ &= \int_{-\infty}^{+\infty} dx^+ \lambda x^+ \frac{M}{\lambda x_0^+} \delta(x^+ - x_0^+) = M . \end{aligned}$$

Hence, the energy carried by the massless field is M ; by energy conservation, it is also equal to the mass of the two dimensional black hole; this can be verified by calculating the Arnowitt-Deser-Misner (ADM) mass of the black hole as done in [10].

4.2 The Trace Anomaly and Hawking Radiation

Classically, the trace of the matter stress-energy tensor is zero since

$$\begin{aligned} T_{ab}^f &= \frac{1}{2} \left(\nabla_a f \nabla_b f - \frac{1}{2} (\nabla f)^2 g_{ab} \right) \\ \Rightarrow T^a_a &= \frac{1}{2} \left((\nabla f)^2 - (\nabla f)^2 \right) = 0 . \end{aligned}$$

(we suppress the label f in the following).

This reflects the conformal invariance of the matter action: under a change

$$g_{ab} \rightarrow e^{2\alpha} g_{ab} ,$$

which, only in two dimensions, also implies

$$\sqrt{-g} \rightarrow e^{2\alpha} \sqrt{-g} ,$$

the matter action

$$S_m = -\frac{1}{2\pi} \int d^2x \sqrt{-g} g^{ab} \frac{1}{2} \nabla_a f \nabla_b f ,$$

changes to

$$S'_m = -\frac{1}{2\pi} \int d^2x \left(e^{2\alpha} \sqrt{-g} \right) \left(e^{-2\alpha} g^{ab} \right) \frac{1}{2} \nabla_a f \nabla_b f = S_m .$$

On the other hand, for an infinitesimal change $\delta g_{ab} = 2\alpha g_{ab}$, the change in S_m is

$$\delta S_m = -\frac{1}{2\pi} \int d^2x \sqrt{-g} T_{ab} \delta g^{ab} = \frac{1}{2\pi} \int d^2x \sqrt{-g} T^a_a 2\alpha ,$$

which vanishes only if $T^a_a \equiv 0$.

In curved spacetime, however, the conformal invariance is broken by the quantum fluctuations of the matter field(s): the expectation value of the trace of the stress-energy tensor turns out to be proportional to the curvature scalar.

We now show this statement in the limit of a weak curvature (we will follow the derivation of [11, pages 185-188], see also [12, pages 468-472]). For this purpose, we will use the minkowskian null coordinates $x^\pm = t \pm x$ in which

$$ds^2 = -dt^2 + dx^2 = -dx^+ dx^- .$$

After a Wick rotation $t = -i\tau$

$$ds^2 = d\tau^2 + dx^2 = dz d\bar{z} ,$$

where

$$\left\{ \begin{array}{c} z \\ \bar{z} \end{array} \right\} = \tau \pm ix = ix^\pm .$$

For weak curvatures, the matter action is

$$\begin{aligned} S [\eta_{ab} + h_{ab}] &= S_0 [\eta_{ab}] - \frac{1}{2\pi} \int d^2 z T_{ab} h^{ab} \\ &= \frac{1}{2\pi} \int d^2 z \frac{1}{2} (\partial f)^2 - \frac{1}{2\pi} \int d^2 z T_{ab} h^{ab} . \end{aligned}$$

The vacuum expectation value of the stress-energy tensor is

$$\begin{aligned} \langle T_{ab} (z, \bar{z}) \rangle &= \frac{\int [df] e^{-S} T_{ab}(z, \bar{z})}{\int [df] e^{-S}} \\ &= 2\pi \frac{1}{Z[h^{ab}]} \frac{\delta Z[h^{ab}]}{\delta h^{ab}(z, \bar{z})} = 2\pi \frac{\delta W[h^{ab}]}{\delta h^{ab}(z, \bar{z})} , \end{aligned}$$

where we have defined

$$Z[h^{ab}] = \int [df] e^{-S} \equiv Z[0] e^{W[h^{ab}]} .$$

$Z[h^{ab}]$ generates the correlation functions involving stress-energy tensors of the free theory defined by S_0 :

$$\langle T_{a_1 b_1} (z_1, \bar{z}_1) \dots T_{a_n b_n} (z_n, \bar{z}_n) \rangle_0 = (2\pi)^n \frac{\delta^n Z[h]}{\delta h^{a_1 b_1} (z_1, \bar{z}_1) \dots \delta h^{a_n b_n} (z_n, \bar{z}_n)} \Big|_{h=0} .$$

$W[h^{ab}]$ generates the corresponding connected correlators

$$\langle T_{a_1 b_1} (z_1, \bar{z}_1) \dots T_{a_n b_n} (z_n, \bar{z}_n) \rangle_0^{(conn)} = (2\pi)^n \frac{\delta^n W[h]}{\delta h^{a_1 b_1} (z_1, \bar{z}_1) \dots \delta h^{a_n b_n} (z_n, \bar{z}_n)} \Big|_{h=0} .$$

Since we will only use connected correlators, we will suppress the label (*conn*) in what follows.

We study

$$\begin{aligned} \langle T_{ab} (z, \bar{z}) \rangle - \langle T_{ab} (z, \bar{z}) \rangle_0 &= 2\pi \left(\frac{\delta W}{\delta h^{ab}(z, \bar{z})} \Big|_h - \frac{\delta W}{\delta h^{ab}(z, \bar{z})} \Big|_{h=0} \right) \\ &= 2\pi \int d^2 z' \frac{\delta^2 W}{\delta h^{ab}(z, \bar{z}) \delta h^{cd}(z', \bar{z}')} \Big|_{h=0} h^{cd} (z', \bar{z}') \\ &= \frac{1}{2\pi} \int d^2 z' \langle T_{ab} (z, \bar{z}) T_{cd} (z', \bar{z}') \rangle_0 h^{cd} (z', \bar{z}') . \end{aligned}$$

In flat space

$$\left\{ \begin{array}{lcl} T(z) & \equiv & T_{zz}(z) = -\frac{1}{2} : \partial_z f \partial_z f : \\ \overline{T}(\overline{z}) & \equiv & T_{\overline{z}\overline{z}}(\overline{z}) = -\frac{1}{2} : \partial_{\overline{z}} f \partial_{\overline{z}} f : \\ T_{z\overline{z}} & = & 0 . \end{array} \right.$$

As operators, T and \overline{T} depend only on z and \overline{z} , respectively. This follows from the fact that $f(z, \overline{z})$ is a free field:

$$\partial_z \partial_{\overline{z}} f(z, \overline{z}) = 0 ,$$

so

$$f(z, \overline{z}) = f_1(z) + f_2(\overline{z}) .$$

The free ($h = 0$) Euclidean matter action is

$$S_0[\eta_{ab}] = \frac{1}{2\pi} \int d^2 z \frac{1}{2} (\partial f)^2 ,$$

so that the propagator satisfies

$$-\nabla^2 G(z, z') = 2\pi \delta^{(2)}(z - z') ,$$

with solution

$$G(z, \overline{z}; z', \overline{z}') = -\ln |z - z'| = -\frac{1}{2} \ln(z - z') (\overline{z} - \overline{z}') .$$

Since $\langle f f \rangle_0 = G$, we get

$$\left\{ \begin{array}{lcl} \langle \partial_z f \partial_{z'} f \rangle_0 & = & -\frac{1}{2} \frac{1}{(z - z')^2} \\ \langle \partial_{\overline{z}} f \partial_{\overline{z}'} f \rangle_0 & = & -\frac{1}{2} \frac{1}{(\overline{z} - \overline{z}')^2} \\ \langle \partial_z f \partial_{\overline{z}'} f \rangle_0 & = & 0 . \end{array} \right.$$

By Wick theorem

$$\left\{ \begin{array}{l} \langle T(z) T(z') \rangle_0 = \frac{1}{4} \langle : \partial_z f \partial_z f : : \partial_{z'} f \partial_{z'} f : \rangle_0 = \frac{1}{4} 2 (\langle \partial_z f \partial_{z'} f \rangle_0)^2 \\ = \frac{1}{8} \frac{1}{(z-z')^4} \\ \langle \bar{T}(\bar{z}) \bar{T}(\bar{z}') \rangle_0 = \frac{1}{8} \frac{1}{(\bar{z}-\bar{z}')^4} \\ \langle T(z) \bar{T}(\bar{z}') \rangle_0 = 0 . \end{array} \right. \quad (34)$$

We get

$$\begin{aligned} \langle T(z, \bar{z}) \rangle - \langle T(z) \rangle_0 = \langle T(z, \bar{z}) \rangle &= \frac{1}{2\pi} \int d^2 z' \langle T(z) T(z') \rangle_0 h^{zz}(z', \bar{z}') \\ &= \frac{1}{16\pi} \int d^2 z' \frac{h^{zz}(z', \bar{z}')}{(z-z')^4} \Theta(|z-z'|^2 - a^2) . \end{aligned}$$

Here, we have inserted a step function short-distance cutoff to render the integral finite and we have used that T is normal ordered in flat space so that $\langle T \rangle_0 = 0$.

This cutoff will introduce an explicit \bar{z} -dependence in $\langle T(z, \bar{z}) \rangle$:

$$\partial_{\bar{z}} \langle T(z, \bar{z}) \rangle = \frac{1}{16\pi} \int d^2 z' \frac{h^{zz}(z', \bar{z}')}{(z-z')^3} \delta(|z-z'|^2 - a^2) .$$

We Taylor expand $h^{zz}(z', \bar{z}')$ around (z, \bar{z})

$$h^{zz}(z', \bar{z}') = h^{zz}(z, \bar{z}) + (z' - z) \partial_z h^{zz}(z, \bar{z}) + (\bar{z}' - \bar{z}) \partial_{\bar{z}} h^{zz}(z, \bar{z}) + \dots$$

Only the term $(z' - z)^3 \frac{1}{3!} \partial_z^3 h^{zz}(z, \bar{z})$ in the expansion contributes to the integral after the limit $a \rightarrow 0$ has been taken

$$\begin{aligned} \partial_{\bar{z}} \langle T(z, \bar{z}) \rangle &= -\frac{1}{16\pi} \frac{1}{3!} \partial_z^3 h^{zz}(z, \bar{z}) \int_0^{+\infty} \frac{1}{2} dr^2 \delta(r^2 - a^2) \int_0^{2\pi} d\theta \\ &= -\frac{1}{96} \partial_z^3 h^{zz}(z, \bar{z}) . \end{aligned}$$

Also

$$\partial_z \langle \bar{T}(z, \bar{z}) \rangle = -\frac{1}{96} \partial_{\bar{z}}^3 h^{\bar{z}\bar{z}}(z, \bar{z}) .$$

The stress-energy tensor must obey the following three physical principles:

First, energy-momentum conservation requires a divergenceless stress-energy tensor:

$$\partial_{\bar{z}} \langle T(z, \bar{z}) \rangle + \partial_z \langle T_{zz}(z, \bar{z}) \rangle = 0 \quad (35)$$

$$\partial_z \langle \bar{T}(z, \bar{z}) \rangle + \partial_{\bar{z}} \langle T_{z\bar{z}}(z, \bar{z}) \rangle = 0 . \quad (36)$$

Second, its trace

$$\begin{aligned} g^{ab} \langle T_{ab} \rangle &= (\eta^{ab} + h^{ab}) \langle T_{ab} \rangle = 2\eta^{z\bar{z}} \langle T_{z\bar{z}} \rangle + \mathcal{O}(\langle \epsilon \rangle) \\ &= 4 \langle T_{z\bar{z}} \rangle + \mathcal{O}(\langle \epsilon \rangle) , \end{aligned}$$

should be invariant under diffeomorphisms:

$$\delta_\xi h^{ab}(z, \bar{z}) = \partial^a \xi^b(z, \bar{z}) + \partial^b \xi^a(z, \bar{z}) .$$

Finally, it must be symmetric.

It is equivalent to require diffeomorphism invariance of the effective action $W[h]$.

So we try to add local counterterms in T_{ab} so as to respect these physical principles. (35) gives

$$\langle T_{zz}(z, \bar{z}) \rangle = \frac{1}{96} \partial_z^2 h^{zz}(z, \bar{z}) ,$$

and (36) gives

$$\langle T_{z\bar{z}}(z, \bar{z}) \rangle = \frac{1}{96} \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}}(z, \bar{z}) .$$

Symmetry requires

$$\langle T_{zz}(z, \bar{z}) \rangle = \langle T_{z\bar{z}}(z, \bar{z}) \rangle = \frac{1}{96} \left(\partial_z^2 h^{zz}(z, \bar{z}) + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}}(z, \bar{z}) \right) ,$$

which in turn implies

$$\begin{aligned} \langle T(z, \bar{z}) \rangle &= -\frac{1}{96} \partial_{\bar{z}}^{-1} \partial_z^3 h^{zz}(z, \bar{z}) - \frac{1}{96} \partial_{\bar{z}} \partial_z h^{\bar{z}\bar{z}}(z, \bar{z}) \\ &= -\frac{1}{96} \partial_{\bar{z}}^{-1} \partial_z \left(\partial_z^2 h^{zz}(z, \bar{z}) + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}}(z, \bar{z}) \right) , \end{aligned}$$

and

$$\langle \overline{T}(z, \bar{z}) \rangle = -\frac{1}{96} \partial_z^{-1} \partial_{\bar{z}} \left(\partial_z^2 h^{\bar{z}\bar{z}}(z, \bar{z}) + \partial_z^2 h^{zz}(z, \bar{z}) \right) .$$

$T_{z\bar{z}}$ is not diffeomorphism invariant because

$$\delta_\xi \langle T_{z\bar{z}}(z, \bar{z}) \rangle = \frac{1}{96} \left(2 \partial_z^2 \partial^z \xi^z(z, \bar{z}) + 2 \partial_{\bar{z}}^2 \partial^{\bar{z}} \xi^{\bar{z}}(z, \bar{z}) \right) .$$

However

$$\begin{aligned} \delta_\xi \left(\partial_{\bar{z}} \partial_z h^{\bar{z}\bar{z}}(z, \bar{z}) \right) &= \partial_{\bar{z}} \partial_z \partial^{\bar{z}} \xi^z(z, \bar{z}) + \partial_{\bar{z}} \partial_z \partial^z \xi^{\bar{z}}(z, \bar{z}) \\ &= \partial_z^2 \partial^z \xi^z(z, \bar{z}) + \partial_{\bar{z}}^2 \partial^{\bar{z}} \xi^{\bar{z}}(z, \bar{z}) , \end{aligned}$$

so add

$$\frac{1}{96} (-2) \partial_{\bar{z}} \partial_z h^{\bar{z}\bar{z}}(z, \bar{z}) ,$$

to $\langle T_{z\bar{z}} \rangle$

$$\langle T_{z\bar{z}}(z, \bar{z}) \rangle = \frac{1}{96} \left(\partial_z^2 h^{zz}(z, \bar{z}) - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}}(z, \bar{z}) + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}}(z, \bar{z}) \right) ,$$

which implies

$$\langle T(z, \bar{z}) \rangle = -\frac{1}{96} \partial_z^{-1} \partial_{\bar{z}} \left(\partial_z^2 h^{zz}(z, \bar{z}) - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}}(z, \bar{z}) + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}}(z, \bar{z}) \right) ,$$

and

$$\langle \overline{T}(z, \bar{z}) \rangle = -\frac{1}{96} \partial_z^{-1} \partial_{\bar{z}} \left(\partial_z^2 h^{zz}(z, \bar{z}) - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}}(z, \bar{z}) + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}}(z, \bar{z}) \right) .$$

The trace is

$$g^{ab} \langle T_{ab} \rangle = 4 \langle T_{z\bar{z}} \rangle = \frac{1}{24} \left(\partial_z^2 h^{zz} - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}} + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}} \right) .$$

One recognizes the right hand side of the above equation as the curvature scalar to first order in h

$$\begin{aligned} \delta R &= g^{ab} \delta R_{ab} = g_{cd} \nabla^2 \delta g^{cd} - \nabla_a \nabla_b \delta g^{ab} \\ &= \eta_{cd} \eta^{ef} \partial_e \partial_f h^{cd} - \partial_a \partial_b h^{ab} \\ &= 4 \eta_{z\bar{z}} \eta^{z\bar{z}} \partial_z \partial_{\bar{z}} h^{z\bar{z}} - \partial_z^2 h^{zz} - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}} - \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}} \\ &= - \left(\partial_z^2 h^{zz} - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}} + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}} \right) . \end{aligned}$$

So

$$\left\{ \begin{array}{lcl} \langle T_{z\bar{z}} \rangle & = & -\frac{1}{24} R \\ \langle T \rangle & = & \frac{1}{96} \partial_z^{-1} \partial_z R \\ \langle \bar{T} \rangle & = & \frac{1}{96} \partial_z^{-1} \partial_{\bar{z}} R . \end{array} \right.$$

Integrating

$$\langle T_{ab} \rangle = 2\pi \frac{\delta W[h]}{\delta h^{ab}} ,$$

we find

$$\begin{aligned} W[h] &= \frac{1}{2\pi} \left(-\frac{1}{48}\right) \int d^2 z \left(\partial_z^2 h^{zz} - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}} + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}} \right) (4 \partial_z \partial_{\bar{z}})^{-1} \\ &\quad \cdot \left(\partial_z^2 h^{zz} - 2 \partial_z \partial_{\bar{z}} h^{z\bar{z}} + \partial_{\bar{z}}^2 h^{\bar{z}\bar{z}} \right) \\ &= \frac{1}{2\pi} \left(-\frac{1}{48}\right) \int d^2 z R (\nabla^2)^{-1} R . \end{aligned}$$

This effective action is called the Polyakov action (it does not contain any term linear in h because $\langle T \rangle_0 = 0$).

These relations also hold in arbitrary spacetimes [13], i.e., the curvature need not be weak, as we assumed in the derivation above. Thus

$$g^{ab} \langle T_{ab} \rangle = +\frac{1}{24} R , \quad (37)$$

and

$$W[g^{ab}] = \frac{1}{2\pi} \left(+\frac{1}{48}\right) \int d^2 x \sqrt{-g} R (\nabla^2)^{-1} R . \quad (38)$$

In a general spacetime, we can evaluate the stress-energy tensor induced by the Polyakov action in the conformal gauge: one way is to vary the Polyakov action and then to go to conformal gauge. The calculation being quite lengthy, we will proceed as follows: we first evaluate T_{+-} using the conformal anomaly:

$$T_{+-} = \left(2 g^{+-}\right)^{-1} \left(+\frac{1}{24}\right) R = -\frac{1}{12} \partial_+ \partial_- \rho .$$

Then, we determine $T_{\pm\pm}$ by requiring T to be divergenceless:

$$\begin{aligned}
\nabla_{\mp} T_{\pm\pm} + \nabla_{\pm} T_{\mp\pm} &= 0 \\
\Leftrightarrow \partial_{\mp} T_{\pm\pm} + (\partial_{\pm} - \Gamma_{\pm\pm}^{\pm}) T_{\mp\pm} &= 0 \\
\Leftrightarrow \partial_{\mp} T_{\pm\pm} &= \frac{1}{12} (\partial_{\mp} \partial_{\pm}^2 \rho - 2 \partial_{\pm} \rho \partial_{\pm} \partial_{\mp} \rho) \\
&= \frac{1}{12} \partial_{\mp} (\partial_{\pm}^2 \rho - \partial_{\pm} \rho \partial_{\pm} \rho) .
\end{aligned}$$

So

$$T_{\pm\pm} = \frac{1}{12} (\partial_{\pm}^2 \rho - \partial_{\pm} \rho \partial_{\pm} \rho + t_{\pm}(x^{\pm})) .$$

The functions $t_{\pm}(x^{\pm})$ are determined by boundary conditions. They reflect the non-locality of the Polyakov action.

Following [9], we compute this quantum induced stress-energy tensor for the collapsing two dimensional dilaton black hole displayed in fig. 9, with metric given by (33) in Kruskal coordinates. In the asymptotically minkowskian coordinates σ^{\pm} , covering the region outside the black hole in fig. 9, the metric takes the form

$$e^{-2\rho'} = \begin{cases} 1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda\sigma^-} & , \quad x^+ < x_0^+ \\ 1 + \frac{M}{\lambda} e^{-2\lambda\sigma} & , \quad x^+ > x_0^+ . \end{cases} \quad (39)$$

The coordinates σ^{\pm} are thus not minkowskian in the (flat) linear dilaton vacuum, $x^+ < x_0^+$.

In terms of σ^{\pm} , the stress-energy tensor is

$$\begin{cases} T_{\sigma^{\pm}\sigma^{\pm}} &= \frac{1}{12} (\partial_{\sigma^{\pm}}^2 \rho' - \partial_{\sigma^{\pm}} \rho' \partial_{\sigma^{\pm}} \rho' + t_{\sigma^{\pm}}(\sigma^{\pm})) \\ &= -\frac{1}{24} \left(e^{2\rho'} \partial_{\sigma^{\pm}}^2 e^{-2\rho'} - \frac{1}{2} e^{4\rho'} (\partial_{\sigma^{\pm}} e^{-2\rho'})^2 - 2 t_{\sigma^{\pm}}(\sigma^{\pm}) \right) \\ T_{\sigma^+ \sigma^-} &= -\frac{1}{12} \partial_{\sigma^+} \partial_{\sigma^-} \rho' = \frac{1}{24} \left(e^{2\rho'} \partial_{\sigma^+} \partial_{\sigma^-} e^{-2\rho'} - e^{4\rho'} (\partial_{\sigma^+} e^{-2\rho'}) (\partial_{\sigma^-} e^{-2\rho'}) \right) . \end{cases} \quad (40)$$

Since the coordinates σ^\pm are minkowskian at J_R^\pm , $\rho' = 0$ there, and

$$\begin{cases} T_{\sigma^\pm \sigma^\pm} = \frac{1}{12} t_{\sigma^\pm}(\sigma^\pm) \\ T_{\sigma^+ \sigma^-} = 0. \end{cases}$$

If the incoming quantum state of the matter f is the vacuum state $|0_{in}\rangle$, then T , which should be written $\langle 0_{in}|T|0_{in}\rangle$, vanishes at J_R^- , hence

$$\begin{aligned} t_{\sigma^+}(\sigma^+) &= 0 \\ t_{\sigma^-}(\sigma^- = -\infty) &= 0. \end{aligned}$$

T must also vanish in the linear dilaton region, $x^+ < x_0^+$. Inserting (39) in (40) we obtain

$$\begin{aligned} 0 = T_{\sigma^+ \sigma^+} &= \frac{1}{12} t_{\sigma^+}(\sigma^+) = 0, \quad x^+ < x_0^+ \\ 0 = T_{\sigma^+ \sigma^-} &= 0, \quad x^+ < x_0^+, \end{aligned}$$

and

$$\begin{aligned} 0 = T_{\sigma^- \sigma^-} &= -\frac{1}{24} \left\{ \left(1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda \sigma^-} \right)^{-1} \frac{M}{x_0^+} e^{\lambda \sigma^-} - \frac{1}{2} \left(1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda \sigma^-} \right)^{-2} \right. \\ &\quad \left. * \left(\frac{M}{\lambda x_0^+} e^{\lambda \sigma^-} \right)^2 - 2 t_{\sigma^-}(\sigma^-) \right\} \\ &= -\frac{1}{24} \left\{ -\frac{1}{2} \left[\left(\frac{\left(\frac{M}{\lambda x_0^+} e^{\lambda \sigma^-} \right)}{\left(1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda \sigma^-} \right)} - \lambda \right)^2 - \lambda^2 \right] - 2 t_{\sigma^-}(\sigma^-) \right\} \\ &= -\frac{1}{24} \left\{ \frac{\lambda^2}{2} \left[1 - \left(1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda \sigma^-} \right)^{-2} \right] - 2 t_{\sigma^-}(\sigma^-) \right\} \\ &\Leftrightarrow t_{\sigma^-}(\sigma^-) = \frac{\lambda^2}{4} \left(1 - \left(1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda \sigma^-} \right)^{-2} \right). \end{aligned}$$

The stress-energy tensor is now completely determined; at J_R^+ , it is equal to

$$\begin{cases} T_{\sigma^+\sigma^-} = 0 \\ T_{\sigma^+\sigma^+} = 0 \\ T_{\sigma^-\sigma^-} = \frac{1}{12} t_{\sigma^-}(\sigma^-) = \frac{\lambda^2}{48} \left(1 - \left(1 + \frac{M}{\lambda^2 x_0^+} e^{\lambda\sigma^-} \right)^{-2} \right) . \end{cases}$$

This represents an energy flow in the σ^+ -direction equal to $T_{\sigma^-\sigma^-}$ per unit length in the σ^- -direction. In the far past, this flux is zero but it builds up to the constant value $\frac{\lambda^2}{48}$ as the horizon is approached: this is the Hawking radiation. We note that the flux is independent of the mass of the black hole unlike the four dimensional Schwarzschild black hole where the flux is found to be inversely proportional to the square of the mass.

As we have shown in section 3, the temperature of a black hole is $\frac{\kappa}{2\pi}$, where κ is the surface gravity of the black hole defined by

$$\kappa = \lim_H \sqrt{(\nabla V)^2} ,$$

where V^2 is the redshift factor

$$V^2 = -\xi_a \xi^a ,$$

ξ^a being the time translation Killing field of the black hole.

The metric of the eternal two dimensional black hole of fig. 7 was given by:

$$ds^2 = \left(1 + \frac{M}{\lambda} e^{-2\lambda\sigma} \right)^{-1} (-d\tau^2 + d\sigma^2) .$$

Thus, $\xi^a = \left(\frac{\partial}{\partial \tau} \right)^a$, as we showed before, and

$$V^2 = -g_{ab} \xi^a \xi^b = -g_{\tau\tau} = \left(1 + \frac{M}{\lambda} e^{-2\lambda\sigma} \right)^{-1} .$$

Then

$$\begin{aligned} \kappa &= \lim_{\sigma \rightarrow -\infty} \sqrt{g^{\sigma\sigma} (\partial_\sigma V)^2} = \lim_{\sigma \rightarrow -\infty} \sqrt{g^{\sigma\sigma}} |\partial_\sigma V| \\ &= \lim_{\sigma \rightarrow -\infty} \frac{1}{\sqrt{g_{\sigma\sigma}}} \frac{1}{2V} |\partial_\sigma V^2| = \lim_{\sigma \rightarrow -\infty} \frac{1}{2V^2} |\partial_\sigma V^2| \\ &= \lim_{\sigma \rightarrow -\infty} \frac{1}{2} \left(1 + \frac{M}{\lambda} e^{-2\lambda\sigma} \right) \left| - \left(1 + \frac{M}{\lambda} e^{-2\lambda\sigma} \right)^{-2} \frac{M}{\lambda} (-2\lambda) e^{-2\lambda\sigma} \right| \\ &= \lim_{\sigma \rightarrow -\infty} \left(1 + \frac{M}{\lambda} e^{-2\lambda\sigma} \right)^{-1} \frac{M}{\lambda} e^{-2\lambda\sigma} \lambda = \lambda . \end{aligned} \tag{41}$$

So the temperature of the two dimensional dilaton black hole is $\frac{\lambda}{2\pi}$.

We can check that the outgoing radiation is thermal by canonically quantizing the matter field f in the classical background geometry of the collapsing two dimensional dilaton black hole depicted in fig. 9. Here, we closely follow the corresponding derivation for a Schwarzschild black hole (For more details, see [14]).

The coordinates σ^\pm defined by

$$\begin{cases} \lambda x^+ &= e^{\lambda\sigma^+} \\ \lambda(x^- + \Delta) &= -e^{-\lambda\sigma^-} \end{cases}, \quad \Delta = \frac{M}{\lambda^3 x_0^+},$$

were minkowskian at J_R^\pm .

On the other hand, the coordinates

$$\begin{cases} \lambda x^+ &= e^{\lambda y^+} \\ \lambda x^- &= -\Delta \lambda e^{-\lambda y^-} \end{cases},$$

are minkowskian in the linear dilaton region and in particular at J_L^- .

The transformations between y^- and σ^- are

$$\begin{cases} \sigma^- &= -\frac{1}{\lambda} \ln \left(\lambda \Delta \left(e^{-\lambda y^-} - 1 \right) \right) \\ y^- &= -\frac{1}{\lambda} \ln \left(\frac{1}{\lambda \Delta} e^{-\lambda \sigma^-} + 1 \right), \end{cases} \quad (42)$$

and the horizon is located at $y^- = 0$.

In the present case, there is no back-scattering since f is a free field:

$$\partial_+ \partial_- f = 0.$$

We concentrate on the right moving field modes because they are the ones responsible for the Hawking effect at J_R^+ .

The modes defined by

$$v_\omega(\sigma^-) = \frac{1}{\sqrt{2\omega}} e^{-i\omega\sigma^-},$$

appear to be positive frequency at J_R^+ . The corresponding wave packets

$$v_{jn}(\sigma^-) = \frac{1}{\sqrt{E}} \int_{jE}^{(j+1)E} d\omega e^{\frac{2\pi i \omega n}{E}} v_\omega(\sigma^-) ,$$

have frequencies around $(j + \frac{1}{2})E$ and are peaked around $u = \frac{2\pi n}{E}$ with time spread $\Delta u = \frac{2\pi}{E}$.

Propagating $v_\omega(\sigma^-)$ back to J_L^- and expressing it as a function of y^- , we get

$$v_\omega(\sigma^-(y^-)) = \frac{1}{\sqrt{2\omega}} e^{\frac{i\omega}{\lambda} \ln(\lambda \Delta(e^{-\lambda y^-} - 1))} \Theta(-y^-) ,$$

and close to the horizon, i.e., at late times σ^- ,

$$v_\omega(\sigma^-(y^-)) \approx \frac{1}{\sqrt{2\omega}} e^{\frac{i\omega}{\lambda} \ln(-\lambda^2 \Delta y^-)} , \quad 0 < -\lambda y^- \ll 1 .$$

Relative to J_L^- , the positive frequency modes are

$$u_\omega(y^-) = \frac{1}{\sqrt{2\omega}} e^{-i\omega y^-} ,$$

and $v_\omega(y^-)$ and $v_{jn}(y^-)$ do not appear to be purely positive frequency relative to J_L^- : it can be shown that for late times, i.e., for large n ,

$$\hat{v}_{jn}(-\omega') = -e^{-\frac{\pi\omega_j}{\lambda}} \hat{v}_{jn}(\omega') , \quad \omega' > 0 ,$$

where $\hat{v}_{jn}(\omega')$ is defined by

$$v_{jn}(y^-) = \int_0^{+\infty} d\omega' (\hat{v}_{jn}(\omega') u_{\omega'}(y^-) + \hat{v}_{jn}(-\omega') u_{\omega'}^*(y^-)) .$$

Proceeding in the same way as for the Schwarzschild black hole, we can get the density matrix describing measurements at J_R^+ on a given late time mode $(i) \equiv (jn)$:

$$D_i = \sum_{N=0}^{+\infty} e^{-N \frac{2\pi\omega_i}{\lambda}} |v_i^N\rangle \langle v_i^N| .$$

This is a completely thermal density matrix with temperature $\frac{\lambda}{2\pi}$.

We now calculate

$$\begin{aligned} \langle \psi | T_{\sigma^- \sigma^-}^{(out)}(\sigma^-) | \psi \rangle &= \langle 0_{in} | S^{-1} T_{\sigma^- \sigma^-}^{(out)}(\sigma^-) S | 0_{in} \rangle \\ &= \langle 0_{in} | T_{\sigma^- \sigma^-}^{(in)}(\sigma^-) | 0_{in} \rangle = \langle 0_{in} | \frac{1}{2} \partial_{\sigma^-} f^{(in)}(\sigma^-) \partial_{\sigma^-} f^{(in)}(\sigma^-) | 0_{in} \rangle , \end{aligned}$$

at J_R^+ .

$\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in}$ being divergent, we regularize it by the point splitting method, thus we define:

$$\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} \equiv \lim_{\delta \rightarrow 0} \left\langle \frac{1}{2} \partial_{\sigma^-} f^{(in)} \left(\sigma^- - \frac{1}{2} \delta \right) \partial_{\sigma^-} f^{(in)} \left(\sigma^- + \frac{1}{2} \delta \right) \right\rangle_{in} .$$

Now

$$f^{(in)}(\sigma^-, \sigma^+) = \int_0^{+\infty} d\omega \left\{ a_{\omega}^{(in)} u_{\omega}(y^-(\sigma^-)) + (a_{\omega}^{(in)})^{\dagger} u_{\omega}^*(y^-(\sigma^-)) + \dots \right\} ,$$

where the ellipsis stand for the σ^+ -dependence of f .

Thus

$$\begin{aligned} \langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} &= \frac{1}{4} \int_0^{+\infty} \frac{d\omega}{\omega} \partial_{\sigma^-} u_{\omega}(y^-(\sigma^- - \frac{1}{2} \delta)) \partial_{\sigma^-} u_{\omega}^*(y^-(\sigma^- + \frac{1}{2} \delta)) \\ &= \frac{1}{4} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- - \frac{1}{2} \delta} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- + \frac{1}{2} \delta} \\ &\quad \cdot \int_0^{+\infty} \frac{d\omega}{\omega} \frac{du_{\omega}}{dy^-}(y^-(\sigma^- - \frac{1}{2} \delta)) \frac{du_{\omega}^*}{dy^-}(y^-(\sigma^- + \frac{1}{2} \delta)) \\ &= \frac{1}{4} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- - \frac{1}{2} \delta} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- + \frac{1}{2} \delta} \\ &\quad \cdot \int_0^{+\infty} d\omega \omega u_{\omega}(y^-(\sigma^- - \frac{1}{2} \delta)) u_{\omega}^*(y^-(\sigma^- + \frac{1}{2} \delta)) \\ &= \frac{1}{4} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- - \frac{1}{2} \delta} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- + \frac{1}{2} \delta} \\ &\quad \cdot \int_0^{+\infty} d\omega \omega \exp \left\{ -i\omega \left[y^-(\sigma^- - \frac{1}{2} \delta) - y^-(\sigma^- + \frac{1}{2} \delta) \right] \right\} \\ &= \frac{1}{4} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- - \frac{1}{2} \delta} \left(\frac{dy^-}{d\sigma^-} \right)_{\sigma^- + \frac{1}{2} \delta} \left[y^-(\sigma^- - \frac{1}{2} \delta) - y^-(\sigma^- + \frac{1}{2} \delta) \right]^{-2} . \end{aligned}$$

Taylor expanding around $\delta = 0$ and taking the limit $\delta \rightarrow 0$, we obtain

$$\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} \rightarrow -\frac{1}{4\delta^2} - \frac{1}{24} \left(\frac{y^{-'''}}{y^{-'}} - \frac{3}{2} \left(\frac{y^{-''}}{y^{-'}} \right)^2 \right) ,$$

where the primes denote derivatives with respect to σ^- .

The term inside the parentheses is the so called schwarzian derivative which

can be evaluated by using the explicit form of the transformation $y^-(\sigma^-)$ given by (42). The result is

$$\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} \rightarrow -\frac{1}{4\delta^2} + \frac{\lambda^2}{48} \left(1 - \frac{1}{(1 + \lambda\Delta e^{\lambda\sigma^-})^2} \right).$$

The divergent term we recognize as the expectation value of $T_{\sigma^-\sigma^-}^{(out)}$ in the out-vacuum. Subtracting this term, we obtain

$$\langle T_{\sigma^-\sigma^-}^{(out)}(\sigma^-) \rangle_\psi = \langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} = \frac{\lambda^2}{48} \left(1 - \frac{1}{(1 + \lambda\Delta e^{\lambda\sigma^-})^2} \right). \quad (43)$$

This agrees with the earlier calculation based on the trace anomaly. Furthermore, it shows that the quantum induced stress-energy tensor really arises from the Hawking radiation.

The above result can also be derived by using the conformal properties of the quantized scalar field f (conformal field theory is reviewed in [15] and [11]).

First, the two-point correlators (34) of the stress-energy tensor defines the central charge associated with f ; here $c = 1$.

Second, it is known that the stress-energy tensor transforms with an anomalous term proportional to c times the schwarzian derivative under a conformal transformation (that is, a transformation which preserves the conformal gauge)

$$T_{\sigma^-\sigma^-}(\sigma^-) = \left(\frac{dy^-}{d\sigma^-} \right)^2 T_{y^-y^-}(y^-) - \frac{c}{24} D_{\sigma^-}^S(y^-), \quad (44)$$

where $D_{\sigma^-}^S(y^-)$ is the schwarzian derivative defined above

$$D_{\sigma^-}^S(y^-) = \frac{y^{-''''}}{y^{-\prime}} - \frac{3}{2} \left(\frac{y^{-''}}{y^{-\prime}} \right)^2.$$

In the derivation of this anomalous transformation law [11, pages 182-183], $T_{y^-y^-}(y^-)$ is assumed to be normal-ordered, i.e., $\langle T_{y^-y^-}^{(in)}(y^-) \rangle_{in} = 0$; taking the expectation value of (44), we get

$$\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} = 0 - \frac{1}{24} D_{\sigma^-}^S(y^-) = \frac{\lambda^2}{48} \left(1 - \frac{1}{(1 + \lambda\Delta e^{\lambda\sigma^-})^2} \right),$$

where we have set $c = 1$. This agrees with (43).

Thus, $\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in}$ cannot be normal-ordered; we now know that this is because of the Hawking radiation:

$$\langle T_{\sigma^-\sigma^-}^{(in)}(\sigma^-) \rangle_{in} = \langle T_{\sigma^-\sigma^-}^{(out)}(\sigma^-) \rangle_{\psi} \neq 0 .$$

Therefore, the three methods of calculating the stress-energy tensor arising from the Hawking radiation give the same answer.

4.3 Back Reaction

We now study the effect of the back reaction of the Hawking radiation on the background geometry [16, 17].

In the two dimensional "Einstein" equations, we add the quantum induced stress-energy tensor to the right hand side. In the conformal gauge, the Einstein equation for ρ (or g_{+-}) becomes

$$2e^{-2\Phi} \left(\partial_+ \partial_- \Phi - \left(2\partial_+ \partial_- \Phi - 2\partial_+ \Phi \partial_- \Phi - \frac{1}{2} \lambda^2 e^{2\rho} \right) \right) = -\frac{N}{12} \partial_+ \partial_- \rho ,$$

and the constraints, i.e., the Einstein equations for $g_{\pm\pm}$, are

$$2e^{-2\Phi} \left(\partial_{\pm}^2 \Phi - 2\partial_{\pm} \rho \partial_{\pm} \Phi \right) = \frac{1}{2} \sum_i \partial_{\pm} f_i \partial_{\pm} f_i + \frac{N}{12} \left(\partial_{\pm}^2 \rho - (\partial_{\pm} \rho)^2 + t_{\pm} (x^{\pm}) \right) .$$

The equations for Φ and f_i are unchanged

$$\partial_+ \partial_- f_i = 0$$

$$-\partial_+ \partial_- \rho + 2\partial_+ \partial_- \Phi - 2\partial_+ \Phi \partial_- \Phi - \frac{1}{2} \lambda^2 e^{2\rho} = 0 .$$

The following combinations of the equations for ρ and Φ will be useful

$$\partial_+ \partial_- \Phi = \left(1 - \frac{1}{24} N e^{2\Phi} \right) \partial_+ \partial_- \rho \quad (45)$$

$$2 \left(1 - \frac{1}{12} N e^{2\Phi} \right) \partial_+ \partial_- \Phi = \left(1 - \frac{1}{24} N e^{2\Phi} \right) \left(4\partial_+ \Phi \partial_- \Phi + \lambda^2 e^{2\rho} \right) . \quad (46)$$

As we see from the two dimensional Einstein equations, $e^{2\Phi}$ is the (square of the) gravitational coupling strength and depends on position. Classically, it varies from zero, asymptotically far away from the black hole, to $\frac{\lambda}{M}$, on the horizon. The semiclassical equations are valid as long as $e^{2\Phi}$ is small.

The linear dilaton vacuum remains an exact solution of these semiclassical equations. This is because the quantum induced stress-energy tensor is zero since the curvature itself is zero.

We can try to find the spacetime due to an incoming shell of matter moving

with the velocity of light. In the interior of the shell, the spacetime is flat and in the exterior, it is given by some solution of the semiclassical equations. Unfortunately, these are nonlinear partial differential equations which cannot be made linear, as was the case for the classical equations (see (31)), and for these reasons, they have never been solved in closed form.

However, close to the infall line, $x^+ = x_0^+ + \varepsilon$, the behaviour of the solution can be found if we assume that it matches continuously onto the vacuum across $x^+ = x_0^+$ and that it approaches the classical black hole solution when $x^- \rightarrow -\infty$.

In fact, $\Sigma \equiv \partial_+ \Phi(x_0^+ + \varepsilon, x^-)$ (which is different from $\partial_+ \Phi(x_0^+ - \varepsilon, x^-)$, i.e., $\partial_+ \Phi(x^+, x^-)$ is discontinuous at x_0^+) satisfies (46) which becomes an ordinary differential equation

$$2 \left(1 - \frac{1}{12} N e^{2\Phi}\right) \partial_- \Sigma - 4 \left(1 - \frac{1}{24} N e^{2\Phi}\right) \partial_- \Phi \Sigma = \left(1 - \frac{1}{24} N e^{2\Phi}\right) \lambda^2 e^{2\rho},$$

here

$$e^{-2\Phi} = e^{-2\rho} = -\lambda^2 x_0^+ x^-.$$

One solution of this inhomogeneous equation is a constant equal to

$$\partial_+ \Phi(x_0^+ - \varepsilon, x^-) = -\frac{1}{2x_0^+}.$$

The general solution can then be found by integration

$$\Sigma + \frac{1}{2x_0^+} = \frac{K e^{2\Phi}}{\sqrt{\left(1 - \frac{N}{12} e^{2\Phi}\right)}} = \frac{K}{\sqrt{-\lambda^2 x_0^+ x^-} \sqrt{-\lambda^2 x_0^+ x^- - \frac{N}{12}}}. \quad (47)$$

As $x^- \rightarrow -\infty$,

$$\Sigma \sim -\frac{1}{2x_0^+} + \frac{K}{-\lambda^2 x_0^+ x^-}.$$

The classical solution has

$$\partial_+ \Phi = -\frac{1}{2} \frac{\partial_+ e^{-2\Phi}}{e^{-2\Phi}} = -\frac{1}{2x_0^+} - \frac{\Delta}{2x_0^+ x^-},$$

thus

$$K = \frac{1}{2} \lambda^2 \Delta = \frac{M}{2\lambda x_0^+}.$$

Σ is singular for

$$e^{-2\Phi} = -\lambda^2 x_0^+ x^- = \frac{N}{12} .$$

The singularity is a curvature singularity because

$$\begin{aligned} R &= 8e^{-2\rho} \partial_+ \partial_- \rho = 8e^{-2\rho} \left(1 - \frac{1}{24} N e^{2\Phi}\right)^{-1} \partial_- \Sigma \\ &= 4 \left(1 - \frac{1}{12} N e^{2\Phi}\right)^{-1} (4e^{-2\rho} \partial_- \Phi \Sigma + \lambda^2) , \end{aligned}$$

where we have used (45) and (46). The singularity is behind the classical horizon, where $e^{-2\Phi} = \frac{M}{\lambda}$, if $\frac{M}{\lambda} > \frac{N}{12}$.

We now define the notion of apparent horizon.

In the four dimensional theory, it is a hypersurface on which $\left(\frac{\partial}{\partial r}\right)^a$ (or equivalently $\nabla_a r$) goes from being spacelike to timelike, i.e., it is defined by the condition $(\nabla r)^2 = 0$. Here, the "radius" r is the quantity which gives the area A of the two-spheres, i.e., $r = \sqrt{\frac{A}{4\pi}}$.

For a static black hole, the apparent horizon coincides with the event horizon.

From the point of view of the two dimensional theory, this radius was found to be proportional to $e^{-2\Phi}$; therefore, the apparent horizon is where $(\nabla \Phi)^2 = 0$.

In the present case, the apparent horizon forms when

$$\partial_+ \Phi(x_0^+, x_0^-) = \Sigma(x_0^-) = 0 ,$$

or, from (47)

$$\begin{aligned} 1 &= \frac{M}{\lambda} \frac{e^{2\Phi}}{\sqrt{1 - \frac{N}{12} e^{2\Phi}}} \\ \Leftrightarrow e^{-2\Phi} &= \frac{N}{24} + \sqrt{\left(\frac{N}{24}\right)^2 + \left(\frac{M}{\lambda}\right)^2} \approx \frac{M}{\lambda} + \frac{N}{24} \\ \Leftrightarrow x_0^- &\approx -\frac{M}{\lambda^3 x_0^+} - \frac{N}{24\lambda^2 x_0^+} , \end{aligned}$$

where we have assumed that $\frac{M}{\lambda} \gg N$.

The latter condition means that the quantum corrections are very small where

the apparent horizon forms:

$$Ne^{2\Phi} \approx N \frac{\lambda}{M} \ll 1 .$$

In this case, we can find the slope of the apparent horizon $\hat{x}^- (x^+)$ as follows:

$$0 = \frac{d}{dx^+} \partial_+ \Phi|_{x^- = \hat{x}^-} = \partial_+^2 \Phi + \frac{d\hat{x}^-}{dx^+} \partial_+ \partial_- \Phi ,$$

because by definition $\partial_+ \Phi = 0$ on the apparent horizon.

Using (46), we get

$$\begin{aligned} \frac{d\hat{x}^-}{dx^+} &= \frac{(-2) \left(1 - \frac{1}{12} Ne^{2\Phi}\right) \partial_+^2 \Phi}{\left(1 - \frac{1}{24} Ne^{2\Phi}\right) \lambda^2 e^{2\rho}} \\ &= - \frac{e^{2\Phi-2\rho} \left(1 - \frac{1}{12} Ne^{2\Phi}\right)}{\lambda^2 \left(1 - \frac{1}{24} Ne^{2\Phi}\right)} T_{++} , \end{aligned} \quad (48)$$

where we also have used the constraint equation for g_{++} in the last line.

Classically, T_{++} is zero everywhere except at x_0^+ , and the horizon is a light-like line as expected.

When the quantum fluctuations of the matter fields are taken into account, T_{++} was found to be

$$T_{++} = -\frac{N}{24} \left(e^{2\rho} \partial_+^2 e^{-2\rho} - \frac{1}{2} e^{4\rho} \left(\partial_+ e^{-2\rho} \right)^2 - 2t_+ \right) .$$

As long as $e^{-2\Phi} \gg N$, we can use the classical black hole solution to evaluate the right hand side of (48). Then

$$e^{-2\rho} \approx e^{-2\Phi} \approx \frac{M}{\lambda} - \lambda^2 x^+ (x^- + \Delta) ,$$

and

$$\frac{d\hat{x}^-}{dx^+} \approx \left(-\frac{1}{\lambda^2} \right) \left(-\frac{N}{24} \right) (-2t_+) .$$

t_+ is determined as before by the boundary condition that there is no incoming radiation from J_R^- :

$$\begin{aligned} 0 &= T_{++}(x^+, x^- \rightarrow -\infty) = -\frac{N}{24} \left(0 - \frac{1}{2} \left(\frac{-\lambda^2 x^-}{-\lambda^2 x^+ x^-} \right)^2 - 2t_+ \right) \\ \Rightarrow t_+(x^+) &= -\frac{1}{4} \left(\frac{1}{x^+} \right)^2. \end{aligned}$$

Thus

$$\frac{d\hat{x}^-}{dx^+} \approx \frac{N}{48\lambda^2} \left(\frac{1}{x^+} \right)^2. \quad (49)$$

Since the value of $e^{-2\Phi}$ on the apparent horizon is the mass $m(x^+)$ of the evaporating black hole, this equation is valid as long as $\frac{m(x^+)}{\lambda} \gg N$.

When $\frac{m(x^+)}{\lambda} \sim N$, the back reaction becomes important and we can no longer use the classical black hole solution in evaluating the right hand side of (48), but then the black hole has radiated most of its mass because

$$\frac{m(x_0^+)}{\lambda} = \frac{M}{\lambda} \gg N.$$

Integrating (49), we obtain

$$\hat{x}^- + \Delta_q = -\frac{N}{48\lambda^2} \frac{1}{x^+}.$$

As $x^+ \rightarrow +\infty$, the apparent horizon thus approaches a global horizon at

$$\hat{x}^- = -\Delta_q = x_0^- + \frac{N}{48\lambda^2 x_0^+} = -\Delta - \frac{N}{48\lambda^2 x_0^+}.$$

We can now show that this recession of the apparent horizon corresponds precisely to the Hawking flux

$$\begin{aligned} -dm(\hat{x}^-) &= -d(\lambda e^{-2\Phi}) = -\partial_+ (\lambda e^{-2\Phi}) dx^+ - \partial_- (\lambda e^{-2\Phi}) dx^- \\ &= -\partial_- (\lambda e^{-2\Phi}) dx^-. \end{aligned}$$

In terms of the asymptotically minkowskian coordinates

$$\begin{aligned} \begin{cases} \lambda x^+ &= e^{\lambda \sigma^+} \\ \lambda (x^- + \Delta_q) &= -e^{-\lambda \sigma^-} \end{cases} , \\ d(\lambda x^-) &= e^{-\lambda \sigma^-} d(\lambda \sigma^-) = -\lambda (x^- + \Delta_q) d(\lambda \sigma^-) \\ &= \frac{N}{48\lambda x^+} d(\lambda \sigma^-) . \end{aligned}$$

Furthermore

$$\partial_- e^{-2\Phi} \approx -\lambda^2 x^+ .$$

So

$$-dm(\sigma^-) = \lambda (\lambda x^+) \frac{N}{48\lambda x^+} d(\lambda \sigma^-) = \frac{N\lambda^2}{48} d\sigma^- .$$

Thus the black hole loses mass at the same rate as it Hawking radiates which is intuitively obvious but very difficult to show for e.g., a four dimensional Schwarzschild black hole.

So far, we have only taken into account the quantum corrections coming from the conformal anomaly of the matter fields. In the large N limit, they dominate over the quantum corrections of the dilaton field and of the conformal factor, but one would like to consider a more systematic quantization of the classical CGHS model defined by the action S_{CGHS} given by (26). The corresponding quantum theory is formally defined by the path integral

$$Z = \int \mathcal{D}(\{, \oplus, \{ \}) \rfloor^{\mathcal{S}_{CGHS}} .$$

The general coordinate invariance allows us to gauge-fix the metric so that it takes the form

$$g_{ab} = e^{2\rho} \hat{g}_{ab} ,$$

where \hat{g}_{ab} is a given reference metric which can be chosen to be the Minkowski metric.

The gauge fixing gives rise to Fadeev-Popov ghosts, hence, the original path integral over g is replaced by an integral over the conformal factor ρ and over the ghosts.

To define the theory, we must find the dependence of the measures in the path integral on ρ . This yields a renormalized or effective action, the renormalization depending on ρ and also on Φ since $e^{2\Phi}$ is the coupling strength of the theory.

The effective action should satisfy the following physical requirements:

First, it should not depend on the arbitrarily chosen reference metric \hat{g} . This can be achieved by requiring the action to be both diffeomorphism invariant with respect to \hat{g} and conformally invariant, that is, invariant under the transformation $\hat{g} \rightarrow e^{2\alpha}\hat{g}$. The latter condition is equivalent to imposing that the total central c vanishes.

Second, as $e^{2\Phi} \rightarrow 0$, the action should reduce to the classical CGHS action and in addition, the leading order corrections in powers of $e^{2\Phi}$ should describe a radiating black hole with a radiation flux proportional to the number N of matter fields.

Moreover, it is desirable that the semiclassical equations are solvable.

A more detailed review on the quantization of the CGHS model is given in [18, Section 4], see also [19, 20, 21].

Several models satisfying the above requirements have been found [19, 20, 21, 22, 23, 24], but here, we will only describe the one proposed by Russo, Susskind and Thorlacius [23, 24].

These authors included a second one-loop term in addition to the Polyakov action in a way that preserves a global symmetry present in the classical CGHS action:

$$S = S_{CGHS} - \frac{\kappa}{8\pi} \int d^2x \sqrt{-g} \left(R \left(\nabla^2 \right)^{-1} R + 2 \Phi R \right) ,$$

S_{CGHS} is the classical CGHS action, and $\kappa = \frac{N}{12}$.

One should also add the contributions from the ghosts, ρ and Φ , to the conformal anomaly but we will not elaborate this further. We only state that it is possible to do this in such a way that $c = 0$ and without affecting our final results, see [25].

In the conformal gauge, the action is

$$\begin{aligned}
S &= \frac{1}{\pi} \int d^2x \left(e^{-2\Phi} (2 \partial_+ \partial_- \rho - 4 \partial_+ \Phi \partial_- \Phi + \lambda^2 e^{2\rho}) + \frac{1}{2} \partial_+ f_i \partial_- f_i \right) \\
&- \frac{\kappa}{\pi} \int d^2x (\partial_+ \rho \partial_- \rho + \Phi \partial_+ \partial_- \rho) .
\end{aligned}$$

The constraints obtained from this action are

$$\begin{aligned}
0 &= \left(e^{-2\Phi} + \frac{\kappa}{4} \right) \left(4 \partial_{\pm} \rho \partial_{\pm} \Phi - 2 \partial_{\pm}^2 \Phi \right) + \frac{1}{2} \partial_{\pm} f_i \partial_{\pm} f_i \\
&- \kappa \left(\partial_{\pm} \rho \partial_{\pm} \rho - \partial_{\pm}^2 \rho - t_{\pm} \right) ,
\end{aligned}$$

and the conserved current corresponding to the global symmetry is

$$j_a = \partial_a (\rho - \Phi) ,$$

with

$$\partial^a j_a = \partial^a \partial_a (\rho - \Phi) = 0 ,$$

as in the classical theory, we can choose coordinates, called the Kruskal coordinates, in which $\rho = \Phi$.

The action can be rewritten as

$$S = \frac{1}{\pi} \int d^2x \left(\partial_+ \left(\kappa \rho + 2e^{-2\Phi} \right) \partial_- (\Phi - \rho) + \lambda^2 e^{2(\rho - \Phi)} + \frac{1}{2} \partial_+ f_i \partial_- f_i \right) .$$

This form suggests the definitions

$$\begin{cases} \kappa \rho + 2e^{-2\Phi} &= \sqrt{\kappa} (\Omega + \chi) \\ \Phi - \rho &= \frac{1}{\sqrt{\kappa}} (\Omega - \chi) , \end{cases}$$

or

$$\begin{cases} \Omega &= \frac{\sqrt{\kappa}}{2} \Phi + \frac{e^{-2\Phi}}{\sqrt{\kappa}} \\ \chi &= \sqrt{\kappa} \rho - \frac{\sqrt{\kappa}}{2} \Phi + \frac{e^{-2\Phi}}{\sqrt{\kappa}} . \end{cases}$$

So

$$S = \frac{1}{\pi} \int d^2x \left(-\partial_+ (\chi - \Omega) \partial_- (\chi + \Omega) + \lambda^2 e^{\frac{2}{\sqrt{\kappa}}(\chi - \Omega)} + \frac{1}{2} \partial_+ f_i \partial_- f_i \right) ,$$

and the constraints become

$$-\kappa t_{\pm} = -\partial_{\pm} (\chi - \Omega) \partial_{\pm} (\chi + \Omega) + \sqrt{\kappa} \partial_{\pm}^2 \chi + \frac{1}{2} \partial_{\pm} f_i \partial_{\pm} f_i .$$

The equations of motion for $(\chi - \Omega)$ resp. $(\chi + \Omega)$ are

$$\begin{cases} -\partial_- \partial_+ (\chi - \Omega) &= 0 \\ -\partial_+ \partial_- (\chi + \Omega) &= \frac{2\lambda^2}{\sqrt{\kappa}} e^{\frac{2}{\sqrt{\kappa}}(\chi - \Omega)} . \end{cases}$$

In Kruskal coordinates

$$\Omega - \chi = \sqrt{\kappa} (\Phi - \rho) = 0 ,$$

and

$$\partial_+ \partial_- \Omega = \partial_+ \partial_- \chi = -\frac{\lambda^2}{\sqrt{\kappa}} .$$

The solutions describing asymptotically flat static geometries are

$$\Omega = \chi = \frac{-\lambda^2 x^+ x^-}{\sqrt{\kappa}} + P \sqrt{\kappa} \ln \left(-\lambda^2 x^+ x^- \right) + \frac{M}{\lambda \sqrt{\kappa}} . \quad (50)$$

The linear dilaton vacuum is still an exact solution with $P = -\frac{1}{4}$ and $M = 0$.

The solution describing a collapsing shell of matter is

$$\begin{aligned} \Omega &= \chi = \frac{-\lambda^2 x^+ x^-}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{4} \ln \left(-\lambda^2 x^+ x^- \right) \\ &\quad - \frac{M}{\lambda \sqrt{\kappa} x_0^+} \left(x^+ - x_0^+ \right) \Theta \left(x^+ - x_0^+ \right) . \end{aligned} \quad (51)$$

As is shown in [23], a curvature singularity forms on the infall line when

$$\begin{aligned} \Omega'(\Phi) &= \frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}} e^{-2\Phi} = 0 \\ \Leftrightarrow e^{-2\Phi} &= \frac{\kappa}{4}, \quad \Omega = \frac{\sqrt{\kappa}}{4} \left(1 - \ln \frac{\kappa}{4} \right) . \end{aligned}$$

For $x^+ > x_0^+$, the curvature singularity lies on the critical line $\Omega = \Omega_{cr}$ defined by

$$\frac{\sqrt{\kappa}}{4} \left(1 - \ln \frac{\kappa}{4} \right) = \frac{-\lambda^2 x^+ (x^- + \Delta)}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{4} \ln (-\lambda^2 x^+ x^-) + \frac{M}{\lambda \sqrt{\kappa}} .$$

The critical line initially lies behind an apparent horizon on which

$$0 = \partial_+ \Omega (= \Omega' \partial_+ \Phi) = -\frac{\lambda^2}{\sqrt{\kappa}} (x^- + \Delta) - \frac{\sqrt{\kappa}}{4} \frac{1}{x^+} ,$$

or

$$-\lambda^2 x^+ (x^- + \Delta) = \frac{\kappa}{4} .$$

Therefore, the apparent horizon recedes at a rate

$$\frac{dx^-}{dx^+} = \frac{\kappa}{4\lambda^2 (x^+)^2} .$$

As in the original CGHS model, this rate of recession corresponds to the Hawking flux.

The critical line and the apparent horizon

$$\begin{cases} \frac{\kappa}{4} \left(1 - \ln \frac{\kappa}{4} \right) &= -\lambda^2 x^+ (x^- + \Delta) - \frac{\kappa}{4} \ln (-\lambda^2 x^+ x^-) + \frac{M}{\lambda} \\ -\lambda^2 x^+ (x^- + \Delta) &= \frac{\kappa}{4} , \end{cases}$$

meet when

$$\begin{cases} -\lambda^2 x^+ (x^- + \Delta) &= \frac{\kappa}{4} \\ -\lambda^2 x^+ x^- &= \frac{\kappa}{4} e^{\frac{4M}{\kappa\lambda}} . \end{cases}$$

Thus

$$\begin{cases} x_s^+ &= x_0^+ \frac{\kappa\lambda}{4M} \left(e^{\frac{4M}{\kappa\lambda}} - 1 \right) \\ x_s^- &= -\frac{\Delta}{1 - e^{-\frac{4M}{\kappa\lambda}}} . \end{cases}$$

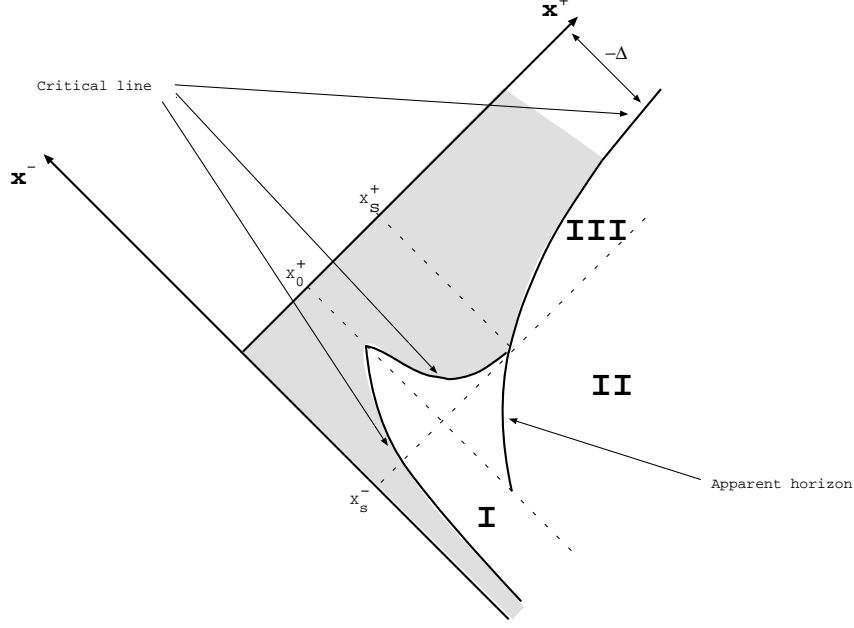


Figure 10:

At this point, the curvature on the apparent horizon is infinite: on the apparent horizon, the curvature is given by

$$R = \frac{4\lambda^2}{(1 - \frac{\kappa}{4} e^{2\Phi})} \rightarrow +\infty \quad \text{as} \quad e^{-2\Phi} \rightarrow e^{-2\Phi_{cr}} = \frac{\kappa}{4} .$$

The critical line, which was spacelike before it met the apparent horizon (since it was behind the apparent horizon), becomes timelike (now it is in front of the apparent horizon).

The spacetime diagram is shown in fig. 10.

It can be shown [23, 24] that if we modify Ω in region *III* by adding a function of x^- (so that the g_{++} -constraints still is satisfied) in such a way that the curves $\Omega = \Omega_{cr}$ and $\partial_+ \Omega = 0$ merge, i.e., we impose the boundary condition $\partial_+ \Omega|_{\Omega=\Omega_{cr}} = 0$ (which also implies $\partial_- \Omega|_{\Omega=\Omega_{cr}} = 0$), then the curvature will be finite on the critical line which can thus be viewed as the boundary of spacetime (analogous to the boundary $r = 0$).

In region *III*, the modified solution turns out to describe a shifted dilaton vacuum:

$$e^{-2\Phi} = e^{-2\rho} = -\lambda^2 x^+ (x^- + \Delta) .$$

Now

$$\begin{aligned}
\Omega &= -\frac{\lambda^2}{\sqrt{\kappa}} x^+ (x_s^- + \Delta) - \frac{\sqrt{\kappa}}{4} \ln(-\lambda^2 x^+ x_s^-) + \frac{M}{\lambda\sqrt{\kappa}} \\
&= -\frac{\lambda^2}{\sqrt{\kappa}} x^+ (x_s^- + \Delta) - \frac{\sqrt{\kappa}}{4} \left(\ln(-\lambda^2 x^+ x_s^-) - \frac{4M}{\lambda\kappa} \right) \\
&= -\frac{\lambda^2}{\sqrt{\kappa}} x^+ (x_s^- + \Delta) - \frac{\sqrt{\kappa}}{4} \ln(-\lambda^2 x^+ (x_s^- + \Delta)) ,
\end{aligned}$$

where we used

$$1 + \frac{\Delta}{x_s^-} = e^{-\frac{4M}{\kappa\lambda}} .$$

Therefore, the solution is continuous across $x^- = x_s^-$. Evaluating $\partial_- \Omega$, we find that its change across $x^- = x_s^-$ is

$$-\frac{\sqrt{\kappa}}{4} \left(\frac{1}{x_s^- + \Delta} - \frac{1}{x_s^-} \right) = \frac{\sqrt{\kappa}}{4} \frac{\Delta}{x_s^- (x_s^- + \Delta)} .$$

This corresponds to a shock wave

$$\begin{aligned}
T_{--}^f &= \frac{1}{2} \partial_- f_i \partial_- f_i = -\sqrt{\kappa} \partial_-^2 \Omega = \frac{\kappa}{4} \frac{\Delta}{x_s^- (x_s^- + \Delta)} \delta(x^- - x_s^-) \\
&= -\frac{\kappa\lambda}{4} \left(1 - e^{-\frac{4M}{\kappa\lambda}} \right) (-\lambda (x_s^- + \Delta))^{-1} \delta(x^- - x_s^-) .
\end{aligned}$$

This shock wave carries a small amount of negative energy $-\frac{\kappa\lambda}{4} \left(1 - e^{-\frac{4M}{\kappa\lambda}} \right)$ out to infinity.

In this model therefore, the black hole evaporates completely.

5 Conclusions

As we have seen in section 3, Hawking's calculation of the black hole radiance yields radiation in a mixed thermal state. For a "superobserver", who is able to see the quantum state ψ of the total Fock space $\mathcal{F}_{\mathcal{H}^+} \otimes \mathcal{F}_{\mathcal{J}^+}$, the state is pure but from the point of view of an observer outside the black hole, who can only access $\mathcal{F}_{\mathcal{J}^+}$, the same state is viewed as a mixed state.

Most of the information contained in the incoming state will end up behind the event horizon of the black hole. This information which from the point of view of the external observer is lost, can be quantified by calculating the so-called entropy of entanglement which is given by

$$E = -\text{tr } \rho \ln \rho ,$$

where ρ is the density matrix describing the mixed thermal state at future null infinity. If ρ had arisen from a pure state, the entropy would have been zero but in the case of a mixed state, it is positive.

As we saw in section 3, the density matrix is obtained from ψ by "tracing over" all horizon states but it can also be obtained directly by applying the so-called superscattering operator $\$$ on the in-state; $\$$ is obtained from the S -matrix by tracing over the horizon states

$$\$ = \text{tr}_{H^+} S S^\dagger .$$

We recall that the calculation of ρ is valid only in the semiclassical regime, when the gravitational field of the collapsing star can be described by a classical real-valued metric. Therefore, the calculation breaks down when the mass of the black hole approaches the Planck mass because at this point, the quantum fluctuations of the metric are expected to become important and of course, the theory describing the quantum fluctuations at this scale is unknown.

We assume for the moment that the unknown planckian dynamics cause the black hole to evaporate completely *and* that the information disappears with the black hole, i.e., the out-state is still described by a mixed state even after the evaporation. In this case, the evolution from in-state to out-state is governed by the superscattering operator $\$$ rather than by a unitary S -matrix; in other words, the usual rules of quantum mechanics are modified in the presence of black holes. This very radical proposal was made by Hawking in 1976 [5]. Since then, however, several authors have argued that this $\$$ -matrix evolution would violate energy conservation and destabilize the vacuum, see [26, 27].

Another possibility is that the planckian physics shuts off the Hawking radiation when the black hole reaches the Planck mass. Hence, the information is stored forever in a Planck-mass remnant, and quantum mechanics is not violated. The main criticism against this scenario is that it is difficult to believe in the stability of the remnant in the absence of any conservation law.

A third proposal is that the information is encoded in the Hawking radiation which therefore is in a pure state.

One possibility is that the information is reemitted during the whole lifetime of the black hole, in direct contradiction with Hawking's calculation. Naively, it seems impossible to realize this idea because the information cannot be reemitted to future null infinity and at the same time go to the black hole: if we assume that a pure in-state evolves to an out-state which is a product state, then a superposition of in-states would evolve to a mixed state; hence, the assumption that the out-state is always a product state is not consistent with the superposition principle. One says that information duplication is not possible in quantum mechanics.

However, the above argument against unitary evolution has been criticized by Susskind who postulates that there is no combined quantum description for a freely falling observer approaching the horizon and a distant observer who remains outside the black hole. Quantum mechanics is valid for each of them separately and in particular, for the external observer, the evolution of quantum states is unitary. Contradictions would certainly arise if the external observer and the freely falling observer could compare their experiments but they cannot, since the freely falling observer disappears into the black hole; the two descriptions are *complementary*. This principle of black hole complementarity is the basis of the approach by Susskind [28, 29]. This principle implies that Planck scale physics enter in the description of the distant observer during the whole process of the evaporation of the black hole, not just in its final stage, as is assumed in Hawking's calculation. This is because the redshift between a point close to the horizon and infinity is enormous.

Susskind has proposed to describe a black hole in terms of a membrane just outside the event horizon (its area is defined to be one Planck unit larger than the area of the event horizon), the so-called stretched horizon, which would absorb and reemit all the information contained in the incoming state [28]. The membrane description had already been useful for classical black holes, see the references in [28]. From the point of view of the distant observer, the stretched horizon behaves like a real physical membrane. In particular, its temperature is so high (of the order of the Planck temperature), that a freely falling observer is seen to be destroyed; but the freely falling observer sees himself passing through

the event horizon without problem, since from his point of view, the stretched horizon does not even exist!

Susskind [30, 31], and before him 't Hooft [32], have speculated that the degrees of freedom of the stretched horizon are described by string theory but it is not known if string theory really provides a mechanism by which the stretched horizon can reemit the original information encrypted in the outgoing radiation, although there are some indications in favor of such a possibility.

Still another possibility is that the information is reemitted after the black hole mass has reached the Planck mass; the restoration of information is then governed by planckian dynamics which we know nothing about. However, general arguments indicate that the information is reradiated very slowly [33]: a four dimensional black hole with initial mass M evaporates down to Planck size in time M^3 , but the decay time for the Planck-size remnant is much longer, at least M^4 , which for a mass of the order of the sun mass is an enormously large number ($10^{108} s$!?!). The problem is again how to explain such a very long decay time, so we are back to the case of an absolutely stable remnant.

Have the two dimensional models resolved the information paradox?

In the RST model, the information does not come back, at least as long as the semiclassical equations are valid. Hence, if the information comes back at all, it is when the black hole reaches the analog of the Planck mass and the result would be a long-lived remnant. Unfortunately, there are other versions of two dimensional gravity which are claimed not to lead to information loss [34, 35] so even in two dimensions, the information paradox has not been resolved. In fact, the physical relevance of two dimensional models has been questioned: it seems likely that it is not possible to truncate the four dimensional theory to include only spherically symmetric modes [36].

One possible resolution of the black hole information paradox is simply that black holes do not exist. One should perhaps look for alternatives to general relativity, for example the recently proposed non-symmetric gravitational theory [37]. The static spherically symmetric solution of this theory, corresponding to the Schwarzschild solution, is everywhere regular and does not contain any event horizon. Black holes are replaced by superdense objects that are stable for arbitrary large masses, due to a new repulsive force that counterbalances the usual attractive gravitational force. Moreover, by choosing a particular parameter entering in the solution to be sufficiently small, the theory agrees with general relativity for the scales where it offers a good description of the experimental data.

Even if this new theory does not survive, it remains an open possibility that black holes do not exist, because they still have to be identified with certainty.

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